HW#11 —Phys410—Fall 2011 Prof. Ted Jacobson
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S11.1 Harmonic oscillator using complex phase space coordiantes

The Hamiltonian for a simple harmonic oscillator is $H = p^2/2 + \omega^2 x^2/2$ in units where the mass $m = 1$. Let a be the complex phase space coordinate $a = \sqrt{\omega/2}(x + ip/\omega)$, and let a^* be its complex conjugate. (a) Express H in terms of a and a^*. (b) Evaluate the Poisson bracket $\{a, a^*\}$, and use that to evaluate $\{a, H\}$ and $\{a^*, H\}$. (c) Write and solve the equations of motion for a and a^* using the Poisson bracket form of Hamilton's equations.

S11.2 Particle in a box with moving wall

A particle of mass m moves in one dimension x between rigid walls at $x = 0$ and $x = \ell$. (a) Using elementary mechanics, show that the average (outward) force on one of the walls is $2E/\ell$, where E is the (kinetic) energy of the particle. (b) Suppose now that the wall at $x = \ell$ is moved adiabatically. The energy of the particle then changes as a result of its collisions with the moving wall. Find the relation between δE and $\delta \ell$, and use this to show that $E\ell^2$ is an adiabatic invariant. (c) Derive the same result instead using adiabatic invariance if $\oint p\, dx$. (Note that you could also find the invariant by dimensional analysis: there is a unique combination of E, ℓ and m with dimensions of action.)

S11.3 Adiabatic changes of a monomial potential

Consider a particle of mass m moving in one dimension in a potential λx^n, where $n \geq 2$ is an even integer, and $\lambda = \lambda(t)$ is a slowly varying function of time.

a) Write out the condition for $\lambda(t)$ to be an adiabatic change, in terms of $\dot{\lambda}$, λ, and some given period T. This defines an "adiabatic parameter" a, proportional to $\dot{\lambda}$, that must satisfy $a \ll 1$ if the adiabatic condition is to hold.

b) Using dimensional analysis, or writing out $\oint dt = \oint dx/v(E, x)$, show that the period is given by $T(E, \lambda) \propto E^{\frac{1}{n}-\frac{1}{2}}/\lambda^{\frac{1}{n}}$.

c) Using dimensional analysis, or writing out $\oint p\, dq$, show that the combination $E^{\frac{1}{n}+\frac{1}{2}}/\lambda^{\frac{1}{n}}$, is an adiabatic invariant. Show that this also implies that $E/\lambda^{\frac{2}{2+n}}$ is invariant.

d) Assuming that λ changes adiabatically and the energy changes accordingly, show that the period satisfies $T \propto 1/E \propto \lambda^{-\frac{2}{2+n}}$.

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S11.4 Numerical study of adiabatic change

Consider a particle of mass $m = 1$ in a potential $V = \lambda x^n$, with $n \geq 2$ an even integer, and $\lambda(t) = (1 - bt)^{-(1+\frac{n}{2})}$. Thus $\lambda(0) = 1$ and $\lambda(1/b) = \infty$, so your simulation should stop before $t = 1/b$, say at $t = 0.9/b$. This form for $\lambda(t)$ is chosen because it will satisfy the adiabatic condition with a constant adiabaticity parameter $a \propto b$, assuming the period evolves adiabatically. Smaller b corresponds to slower changes of λ.

Write a computer code, using any software you like, to evolve the particle from initial conditions, plot the trajectory in phase space, and plot the would-be adiabatic invariant $E/\lambda^{\frac{2}{2+n}}$ you found in the previous problem, for adjustable values of b. Consider at least the two cases $n = 2$ (harmonic oscillator) and $n = 4$ (quartic potential), and produce at least one set of plots for each of these.

Feel free to ignore this, but if you want to find the n dependence of a/b you'll need:

$$\int_{-1}^{1} \frac{ds}{\sqrt{1 - s^n}} = 2\sqrt{\pi}\frac{\Gamma(1 + \frac{1}{n})}{\Gamma(\frac{1}{2} + \frac{1}{n})} \in (2, \pi] \quad \text{for} \quad n = 2, 4, 6, \ldots.$$

S11.5 Cosmological redshift

In S10.3 you worked out the redshift of light or photons climbing out of a gravitational potential well. In cosmology there is also a redshift, due to expansion of the universe, which stretches wavelengths and dilates time intervals. To see how this works, let's adopt the line element

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$$

(see Nov. 10 notes), where $a(t)$ is some function that grows in time and I use units with $c = 1$. (*Suggestion: Draw spacetime diagrams to visualize this situation.*)

1. Consider two wavecrests of light, traveling at the speed of light in the x direction and passing points x_1 and x'_1 respectively at time t_1. Show that the difference $x'_2 - x_2$ of the x coordinates of the two wavecrests when they arrive at time t_2 is equal to $x'_1 - x_1$. (This conservation law results from spatial translation symmetry.)

2. The proper wavelength at t_1 is $\lambda_1 = a(t_1)(x'_1 - x_1)$. If λ_2 is the proper wavelength when the two wavecrests arrive at time t_2, what is λ_2/λ_1? What is the ratio of proper frequencies f_2/f_1?

3. Now let's derive the result in a different way, using time instead of length. Suppose a light pulse is emitted at t_1 in the x direction and travels from x to x', where it arrives at time t_2. (i) Write an equation, involving an integral, showing how the time $t_2 = t_2(\Delta x, t_1)$ is determined by t_1 and $\Delta x = x' - x$. (ii) By taking the derivative of this integral equation with respect to t_1, holding Δx fixed, show that $dt_2/dt_1 = a(t_2)/a(t_1)$. (iii) Show how this recovers the result for the frequency found in part (b), by interpreting dt_1 as the period of an oscillation of a wave emitted around the time t_1. [*Note that this also implies a cosmological time dilation: a process that lasts a proper time dt_1 appears to last a proper time dt_2 when viewed with radiation that travels from t_1 to t_2. This effect has been observed, for example, in flaring light from distant supernovae.*]