Homework 5

Exercise from Thorne / Buonanno lecture at Caltech in 2002

Post-Newtonian Expansion of Waveform from an Inspiraling, Circular Binary with an Extreme Mass Ratio. *Note: This exercise looks very long; actually it is less long than it looks — you are led by the hand through calculations that, in most cases, are rather easy and quick.* Consider a binary consisting of a heavy black hole with mass M orbited by a neutron star with mass $\mu \ll M$, and assume that the spins of the hole and the star are negligible. The black hole's spacetime metric is given by Schwarzschild's formula

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{1}$$

The neutron star moves in a circular geodesic orbit in the equatorial plane $\theta = \pi/2$. The components of the star's 4-momentum are $p^\alpha = m dx^\alpha/d\tau$, where τ is proper time along its orbit.

a. Explain why the orbital angular velocity, as measured by an observer far from the binary, is $\Omega = d\phi/dt = p^\phi/p^t$. One can show that the usual Keplerian formula

$$\Omega = \sqrt{M/r^3} \tag{2}$$

(with r the radius of the orbit) is valid without change (valid fully relativistically) for this (and any) circular geodesic orbit in the Schwarzschild metric; see, e.g., Eq. (11.21) of Schutz, *A First Course in General Relativity* or Exercise 25.19 of Misner, Thorne and Wheeler *Gravitation*.

b. Because the Schwarzschild metric is independent of the time coordinate, the covariant component of the 4-momentum, $p_t \equiv -E$, is a constant of the motion not just for circular geodesic orbits but for any geodesic orbit The quantity E is the conserved energy of the body that moves along the orbit. For the neutron star's circular orbit, use the relations $p^\alpha p^\beta g_{\alpha\beta} = -\mu^2$ (explain where this comes from) and $p^\phi/p^t = \Omega = \sqrt{M/r^2}$ to show that

$$E = \mu \frac{1 - 2M/r}{\sqrt{1 - 3M/r}} . \tag{3}$$

This is an exact relation, not an approximate, post-Newtonian one; but we shall take its post-Newtonian limit below.

c. Note that as $r \to \infty$, $E \to \mu$. This means that E contains the star's rest-mass energy. Show that at large radii, $E \simeq \mu - \mu M/2r$. This is the standard Newtonian formula for the orbital energy: rest mass μ plus kinetic energy equal to $\mu M/2r$ plus gravitational potential energy $-\mu M/r$.

d. Draw a graph of $E(r)$. Notice that it decreases monotonically with decreasing r until $r = 6M$, where it begins increasing. As energy is gradually lost to gravitational waves, the radius will shrink from $r \gg 6M$ to $r = 6M$. Thereafter, further losses of energy cannot be accommodated by circular geodesic orbits. There are no such orbits with energies smaller than that at $r = 6M$. But energy continues to be lost to gravitational waves. What must happen (and does happen) is that the star plunges toward the hole's horizon, on a noncircular orbit, once it reaches $r = 6M$. Thus, $r = 6M$ is the innermost stable circular orbit, *isco*.

e. The orbiting neutron star emits gravitational waves that are predominantly at twice the orbital frequency, $f = 2\Omega/2\pi = \Omega/\pi$, though there are also harmonics at other multiples of Ω/π. For simplicity we shall focus on those waves that come out at this predominant frequency f. Explain why f is the frequency measured by an observer far from the hole, but not the frequency measured near the neutron star.

f. Define the parameter $v \equiv (\pi M f)^{1/3}$. Show that for a circular geodesic orbit at any radius r, $v = \sqrt{M/r}$ is an exact relation. Show that at large radii r, this v is the speed of the star in its orbit. At small radii it is of order that speed, but the exact value of the speed depends on the reference frame of the measurer. Suppose that the measurer is at rest outside the black hole (r, θ, ϕ constant) at a location through which the star's orbit passes. Show that the speed the observer measures as the star whizes by is $v/\sqrt{1 - 2M/r}$.

g. When one uses post-Newtonian techniques to compute the energy carried off by the gravitational waves (the waves' luminosity), one obtains the following formula:

$$\mathcal{F} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}\left[1 - \frac{1247}{336}v^2 + 4\pi v^3 + \mathcal{O}(v^4)\right] \tag{4}$$

Verify that the term preceding the square brackets is the prediction of the quadrupole formula when the orbit is regarded as Newtonian, as derived in Exercise 4 of Week 6. The term $(1247/336)v^2$ is a post-Newtonian correction that includes mass octupole radiation and a variety of other post-Newtonian effects. The post$^{1.5}$-Newtonian term $4\pi v^3$ is produced by the waves' tails — i.e., by that part of the waves that scatters off the black hole's spacetime geometry as it tries to escape from the hole's vicinity, propagates back in toward the hole, then deflects around the hole and reemerges, delayed relative to the prompt waves that carry the Newtonian and post-Newtonian energy. We shall be interested in studying the detectability of this tail contribution to the waves' luminosity.

h. Perform a post-Newtonian expansion of the orbit's energy E to obtain, up to errors of post^2-Newtonian order,

$$E = \mu - \frac{1}{2}\mu v^2\left[1 - \frac{3}{4}v^2 + \mathcal{O}(v^4)\right] \tag{5}$$

i. Show that the law of energy conservation, $dE/dt = -\mathcal{F}$ implies the waves' frequency f, or equivalently $v = (\pi M f)^{1/3}$, evolves with time t (time as measured by observers far from the hole) in the following manner:

$$t(v) = t_{\rm ref} + M\int_v^{v_{\rm ref}} \frac{dE(v')/dv'}{\mathcal{F}(v')}dv' \,. \tag{6}$$

Here $v_f \equiv (\pi M f_{\rm ref})^{1/3}$ is the value of v when some reference frequency (e.g., 100 Hz, or 1000 Hz, or whatever you wish) is reached, and $t_{\rm ref}$ is the time at which that reference frequency is reached. Equation (6) can be thought of as giving the time t_f at which frequency f, corresponding to $v = (\pi M f)^{1/3}$ is reached. *Derive a formula for t_f as a power series in v up through post[1.5]-Newtonian order.* Show that at the leading, Newtonian order your result can be expressed in terms of the binary's chirp mass $\mathcal{M} \equiv \mu^{3/5} M^{2/5}$. Notice that the post-Newtonian and higher-order corrections carry information about the hole's mass M. Therefore, if the waves' frequency evolution were measured, from the Newtonian order result we could infer the chirp mass and then from the higher order corrections we could infer the hole mass M, and knowing $\mathcal{M}$ and M we could infer the neutron star mass μ. We could invert your expansion for t_f to get the frequency f as a power series in time t, if we wished; but we shall not need it below. Rather, in our final result at the end of this problem, we shall need time as a function of frequency, t_f.

j. The waves' phase $\phi = \int 2\pi f dt$ can be thought of equally well as a function of time t, or a function of the frequency f that is reached at time t, or as a function of $v = (\pi M f)^{1/3}$. Show that

$$\phi(f) = \phi_{\rm ref} + 2 \int_v^{v_{\rm ref}} \frac{v'^3 dE(v')/dv'}{\mathcal{F}(v')} dv' \ , \tag{7}$$

where $\phi_{\rm ref}$ is the value of the phase when the reference frequency is reached, and the v's on the right hand side are to be thought of as functions of f, $v = (\pi M f)^{1/3}$.

k. *Derive a formula for $\phi(f)$ as a post-Newtonian expansion in v*, accurate up through post[1.5]-Newtonian order.

l. The post[1.5]-Newtonian term in $\phi(f)$ is the one that arises from the tails of the waves. How many radians of phase does this term contribute, in the LIGO-II frequency band (from about 10 Hz to about 1000 Hz) in the case of a $1.4 M_\odot$ neutron star spiraling into a $10 M_\odot$ black hole? With what accuracy, roughly, would you expect that the influence of the waves' tails can be measured?

m. The gravitational waves measured at Earth will have the form

$$h(t) = A(t) \cos\phi(t) \ , \tag{8}$$

where $\phi(t)$ is the phase computed above, regarded as a function of time t, and where the amplitude $A(t)$ is $A(t) \propto f^{2/3} \propto v^2$, with f and v the values reached at time t. This expression for the amplitude is actually just the Newtonian order term in a post-Newtonian expansion. Since the data analysis is highly sensitive to the waves' phase evolution $\phi(t)$ but not very sensitive to the amplitude evolution, we evaluate A only at leading, Newtonian order while evaluating ϕ to as high an order as our fortitude permits. As we shall see , gravitational wave signal processing is best analyzed using not $h(t)$ but instead its Fourier transform $\tilde{h}(f) = \int_{-\infty}^{+\infty} e^{2\pi i f t} h(t) dt$. Show that $\tilde{h}(-f) = \tilde{h}^*(f)$ where the star denotes complex conjugation. This permits us to restrict attention to positive frequencies. The remainder of this exercise evalutes $\tilde{h}(f)$

n. By evaluating the Fourier transform using the stationary phase approximation, show that

$$\tilde{h}(f) = B(f)e^{i\psi_f - i\pi/4} \tag{9}$$

where the amplitude of the Fourier transform is

$$B(f) \simeq \frac{1}{2}\frac{dt_f}{df}A(t_f) \propto f^{-7/6}\,, \tag{10}$$

and the phase of the Fourier transform, expressed as a function of frequency f, is

$$\psi_f = 2\pi f t_f - \phi(f)\,. \tag{11}$$

l. Use your post-Newtonian expansions for t_f and $\phi(f)$ to obtain the following expansion for the waves' phase

$$\psi_f = 2\pi f t_{\rm ref} - \phi_{\rm ref} + \frac{3}{128}(\pi\mathcal{M}f)^{-5/3}\left[1 + \frac{3715}{756}v^{2/3} - 16\pi v + \mathcal{O}(v^{4/3})\right]\,. \tag{12}$$

Reading:

Damour, Iyer and Sathyaprakash, Phys. Rev. D63 044023 (2001)

Notably:

- the introduction to Sec. II and Sec II A, on the PN-based Taylor expansion of the binary's phase. Tables I and II which give the coefficients in the Taylor expansion.
- Sec III on the frequency domain expansion of the phase.