Homework 4

Solving exercises is the most effective way of learning physics. Although only one third of the final grades for this course will be based on the homeworks, you should take them very seriously.

Assignement to be turned in at the beginning of the class on Thursday, April 20 by students registered to the course:

- Work the three exercises below

Exercises:

Radiation from accelerated masses

We consider the gravitational radiation produced during the elastic deflection of a particle by a fixed scattering center. We denote the initial and final four-momenta of the particle with p^μ and p'^μ. The energy-momentum tensor of the particle is

$$T^{\mu\nu}(\mathbf{x}) = \frac{p^\mu\, p^\nu}{\gamma\, m}\, \delta^{(3)}(\mathbf{x} - \mathbf{x_0}(t))\,, \tag{1}$$

where m is the particle's mass, $\gamma = 1/(1 - v^2/c^2)$ and $\mathbf{x_0}(t)$ is the particle's trajectory. Approximating the collision as instantaneous, the Fourier transform of the energy-momentum tensor for a massive particle, Eq. (1), reads

$$\tilde{T}^{\mu\nu}(\omega, \mathbf{k}) = \frac{c}{i\gamma\, m}\left[\frac{p^\mu\, p^\nu}{\omega - \mathbf{k}\cdot\mathbf{v}} - \frac{p'^\mu\, p'^\nu}{\omega - \mathbf{k}\cdot\mathbf{v}'}\right]. \tag{2}$$

1. **Elastic collisions: non-relativistic limit [M. Maggiore (2006)] (3.5 points)**

   Show that in the non-relativistic limit, Eq. (1) reduces to

   $$\tilde{T}^{ij}(\omega) \simeq -\frac{i\, c}{\omega\, m}\,(p_i\, p_j - p'_i\, p'_j)\,. \tag{3}$$

   Assume $\mathbf{v} = v\,(1, 0, 0)$ and $\mathbf{v}' = v\,(\cos\theta_s, \sin\theta_s, 0)$ and $|\mathbf{v}| = |\mathbf{v}'|$ (elastic collision). In class we derived that the distribution of the energy radiated in gravitational waves is ($d\Omega = d\cos\theta\, d\phi$)

   $$\frac{dE}{d\Omega} = \frac{G}{2\pi^2\, c^7}\,\Lambda_{ij,kl}(\mathbf{n}) \int_0^{+\infty} d\omega\, \omega^2\, \tilde{T}_{ij}(\omega, \omega\,\mathbf{n}/c)\, \tilde{T}^\star_{kl}(\omega, \omega\,\mathbf{n}/c)\,, \tag{4}$$

   where

   $$\mathbf{n} = (\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta)\,, \qquad \Lambda_{ij,kl} = P_{ik}\, P_{jl} - P_{ij}\, P_{kl}/2 \qquad P_{ij} = \delta_{ij} - n_i\, n_j\,. \tag{5}$$

   Plugging Eq. (3) into Eq. (4), derive the following formula

   $$\frac{dE}{d\Omega\, d\omega} = \frac{G\, m^2\, v^4}{\pi^2\, c^5}\left[f_1(\theta_s) - f_2(\theta_s;\phi)\, \sin^2\theta + f_3(\theta_s;\phi)\, \sin^4\theta\right]. \tag{6}$$

   What are the explicit expressions for the coefficients f_1, f_2 and f_3? Show that the radiation is mostly emitted along the z-axis. Evaluate $dE/d\omega$.

2. **Elastic collisions: relativistic limit [M. Maggiore (2006)] (3 points)**

   In the relativistic limit $\mathbf{p} = \gamma m\mathbf{v}$. Setting $\mathbf{k} = \mathbf{n}\,\omega/c$, we have

   $$\tilde{T}^{ij}(\omega, \mathbf{n}\,\omega/c) = \frac{c\,\gamma\, m}{i\omega}\left(\frac{v_i\, v_j}{1 - \mathbf{n}\cdot\mathbf{v}/c} - \frac{v'_i\, v'_j}{1 - \mathbf{n}\cdot\mathbf{v}'/c}\right). \tag{7}$$

Plugging Eq. (7) into Eq. (4), derive the following formula

$$\frac{dE}{d\Omega\, d\omega} = \frac{G\, m^2\, \gamma^2\, v^4}{\pi^2} \left[f_1(v, \theta_s; \theta, \phi) - f_2(v, \theta_s; \theta, \phi) \sin^2\theta + f_3(v, \theta_s; \theta, \phi) \sin^4\theta \right] . \tag{8}$$

What are the explicit expressions for the coefficients f_1, f_2 and f_3? Notice that the factors $1/(1-\mathbf{n}\cdot\mathbf{v}/c)$ appearing in the coefficients f_1, f_2 and f_3 tend to bend the radiation in the direction of the motion.

3. **Lack of beaming from accelerated masses [M. Maggiore (2006)] (4 points)**

Here we want to compare the GW radiation to the EM radiation and show that in the former case there is a lack of beaming.

We consider a relativistic particle which is instantaneously accelerated from 0 to v. The Fourier transform of the energy-momentum tensor reads

$$\tilde{T}^{ij}(\omega, \mathbf{n}\omega/c) = \frac{c\,\gamma\, m}{i\omega} \left(\frac{v_i\, v_j}{1 - \cos\theta\, v/c} \right) . \tag{9}$$

Show that the distribution of the energy radiated in gravitational waves is

$$\left(\frac{dE}{d\Omega\, d\omega} \right)_{\rm GW} = \frac{G\, m^2\, \gamma^2}{4\pi^2 c} \left(\frac{v}{c} \right)^4 p_{\rm GW}(\theta) , \tag{10}$$

and evaluate the coefficient $p_{\rm GW}(\theta)$. The distribution of the energy radiated in electromagnetic waves is

$$\left(\frac{dE}{d\Omega\, d\omega} \right)_{\rm EM} = \frac{e^2}{4\pi^2 c} \left(\frac{v}{c} \right)^2 p_{\rm EM}(\theta) , \qquad p_{\rm EM}(\theta) = \frac{\sin^2\theta}{(1 - \cos\theta\, v/c)^2} . \tag{11}$$

Estimate for which value of θ, the functions $p_{\rm EM}(\theta)$ and $p_{\rm GW}(\theta)$ have a maximum. Estimate also the width of the maximum. From these results you should conclude that even in the limit $\gamma \to \infty$ the gravitational radiation is not beamed in a narrow cone, as the electromagnetic radiation does, but it is extended over a solid angle comparable to 4π.