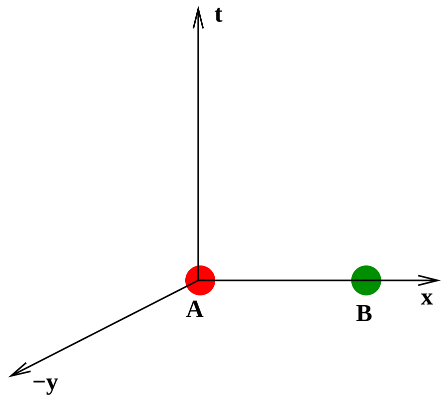

FIG. 1: Before the GW arrives particle A and particle B are at rest along the x-axis and separated by a distance L.

Homework: Lectures 3, 4 and 5

Solving exercises is the most effective way of learning physics. Although only one third of the final grades for this course will be based on the homeworks, you should take them very seriously.

Recommended readings:

1. Chapters 8 and 9 in B. Schutz, " A first course in general relativity".
2. Local Lorentz and free-falling frames, Sec. 8.4 in J. Hartle, " Gravity".
3. Eikonal approximation, see, e.g., Sec. 22.4 and 22.5 in C. Misner, K.S. Thorne and J.A. Wheeler, "Gravitation".
4. Proper detector frame, see, e.g., W. Ni and M. Zimmermann, Phys. Rev. D **17**, 1473 (1978).
5. Transverse-traceless (TT) frame versus local Lorenz frame, see, e.g., M. Rakhmanov Phys. Rev D **71**, 084003 (2005).
6. Newtonian and relativistic tidal gravity, see e.g., Sec. 24.2-24.5 in R.D. Blandford and K.S. Thorne, http://www.pma.caltech.edu/Courses/ph136/ph136.html
7. Interferometer analysed in TT frame and local Lorenz frame, Sec. 26.5 in in R.D. Blandford and K.S. Thorne, http://www.pma.caltech.edu/Courses/ph136/ph136.html

Assignement to be turned in at the beginning of the class on Thursday, February 23 by students registered to the course:

- State what of the above readings you have done
- Work the three exercises below

Exercises:

1. **Interaction of GWs and test particles: transverse-traceless gauge** (3 points)

   Using the geodesic equation, show that in the *transverse-traceless frame* a test particle initially at rest before the gravitational-wave arrives, remains at rest at later times.

   Consider two test particles A and B originally at rest at time $t = 0$, as in Fig. 1. Assume that a monochromatic GW with frequency ω and $+$ polarization (h_+) arrives at time $t > 0$ along the z-axis. Working at linear order in h_+, show that the proper distance between A and B varies as

   $$s = L\,(1 + \frac{1}{2}h_+\,\cos\omega t)\,.$$

2. **Interaction of GWs and test particles: free-falling frame** (4 points)

   Consider two test particles originally at rest at time $t = 0$, as in Fig. 1 and assume that a GW h_{ij} impinges on the test particles at time $t > 0$. The geodesic deviation equation is

   $$\nabla_u\,\nabla_u \xi^\mu = -R^\mu{}_{\nu\rho\sigma}\,\xi^\rho\,u^\nu\,u^\sigma \qquad u^\mu = \frac{dx^\mu}{d\tau} \qquad \xi^\mu = x^\mu_B - x^\mu_A\,.$$

   neglecting terms $\lambda_{\rm GW}/L \ll 1$ ($\lambda_{\rm GW}$ being the gravitational wavelength), working at leading order in h_{ij}, show that in the *free-falling frame* attached to particle A, the geodesic deviation equation reduces to:

   $$\frac{d^2\xi^i}{dt^2} = \frac{1}{2}\ddot{h}^{\rm TT}_{ij}\,\xi^j\,.$$

3. **Geometric optic (eikonal approximation) equation for GW propagation** (3 points)

   In the geometric optic approximation the trace-reversed metric perturbation is written as

   $$\overline{h}_{\mu\nu} = {\rm Re}(\mathcal{A}_{\mu\nu}\,e^{i\phi})$$

   where ϕ is the GW phase which varies on the very short lengthscale $\lambda_{\rm GW}$, and $\mathcal{A}_{\mu\nu}$ is the GW amplitude which varies on the much longer lengthscale $\mathcal{L}$ ($\mathcal{L}$ being the smaller of the radius of curvature of the wavefront and the radius of curvature of spacetime).

   Introduce $k_\mu \equiv \nabla_\mu\phi$ and show that leading order in the small parameter $\lambda_{\rm GW}/\mathcal{L} \ll 1$, the Lorenz gauge condition $\overline{h}^{\mu\nu}{}_{;\nu} = 0$ reduces to the transversality condition $\overline{h}^{\mu\nu}\,k_\nu = 0$.

   Show that at the leading order in $\lambda/\mathcal{L}$, the equation $\overline{h}_{\mu\nu;\alpha}{}^{\alpha} = 0$ reduces to the statement that k_μ is a null vector $k^\mu k_\mu = 0$.