

Observables from Correlation Functions

In this chapter we learn how to compute physical quantities from correlation functions beyond leading order in the perturbative expansion. We will not discuss ultraviolet divergences here; this important subject is left for the next chapter. We will illustrate our results using quantum field theories in lower spacetime dimensions, which do not have ultraviolet divergences.

1 Physical Mass

We begin with the mass of a particle. By itself this is not a very exciting observable, but it is the first step in doing physical calculations beyond leading order in perturbation theory. At tree level, this is just the coefficient of the ϕ^2 term, but at loop level the two are no longer the same.

1.1 Mass Poles

We begin with the 2-point function in scalar field theory. We will show that 1-particle intermediate states give rise to a pole singularity in the 2-point function at the physical mass. This will allow us to give a precise definition of the physical mass.

The operator formulation tells us that

$$\langle \phi(x_1)\phi(x_2) \rangle = \langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle, \quad (1.1)$$

where $\hat{\phi}$ is the Heisenberg field operator, and $|0\rangle$ is the exact vacuum, *i.e.* the ground state of the full interacting Hamiltonian. We write

$$\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \theta(x_1^0 - x_2^0)\langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle + (1 \leftrightarrow 2). \quad (1.2)$$

where

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (1.3)$$

We now insert a complete set of states in each of the time orderings above. The states we use are eigenstates of the full interacting Hamiltonian. These consist of the vacuum, the 1-particle states, 2-particle states, *etc.*:

$$1 = |0\rangle\langle 0| + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\vec{p}\rangle\langle \vec{p}|}_{\text{1-particle states}} + \text{2-particle states} + \dots, \quad (1.4)$$

where

$$E_{\vec{p}} = +\sqrt{\vec{p}^2 + m_{\text{phys}}^2}. \quad (1.5)$$

Note that the physical mass appears because the 1-particle states are the eigenstates of the full Hamiltonian.

Inserting Eq. (1.4) into Eq. (1.2) gives

$$\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \langle 0|\hat{\phi}(x_1)|\vec{p}\rangle \langle \vec{p}|\hat{\phi}(x_2)|0\rangle + (1 \leftrightarrow 2) + \dots \quad (1.6)$$

Here we assume that the vacuum expectation value of the field vanishes, *i.e.*

$$\langle 0|\hat{\phi}(x)|0\rangle = 0. \quad (1.7)$$

As long as we are dealing with a vacuum that is Lorentz invariant and translation invariant, the vacuum expectation value is a constant, and we can work with shifted fields for which Eq. (1.7) is satisfied. The leading contribution therefore arises from the 1-particle states. We will show that these give rise to a pole at the physical mass. We will later show that the contribution from states with 2 or more particles are analytic near $p^2 = m^2$, so the 1-particle states give the complete pole structure.

We can simplify the matrix element by using translation invariance:

$$\begin{aligned} \langle 0|\hat{\phi}(x)|\vec{p}\rangle &= \langle 0|e^{i\hat{P}_\mu x^\mu} \hat{\phi}(0) e^{-i\hat{P}_\mu x^\mu} |\vec{p}\rangle \\ &= \langle 0|\hat{\phi}(0)|\vec{p}\rangle e^{-ip_\mu x^\mu}. \end{aligned} \quad (1.8)$$

Here \hat{P}_μ is the physical 4-momentum operator that acts as the spatial- and time-translation operator on states. In the last line, we have used the translation invariance of the vacuum state to write $\hat{P}|0\rangle = 0$. From now on we leave $p_0 = E_{\vec{p}}$ implicit for brevity. We now use the fact that $\hat{\phi}(0)$ is Lorentz invariant to write

$$\begin{aligned} \langle 0|\hat{\phi}(x)|\vec{p}\rangle &= \langle 0|\hat{U}(L)\hat{\phi}(0)\hat{U}^\dagger(L)|\vec{p}\rangle e^{-ip \cdot x} \\ &= \langle 0|\hat{\phi}(0)|L^{-1}\vec{p}\rangle e^{-ip \cdot x} \\ &= \langle 0|\hat{\phi}(0)|\vec{p}=0\rangle e^{-ip \cdot x} \end{aligned} \quad (1.9)$$

where $\hat{U}(L)$ is the unitary representation of the Lorentz transformation L on the physical states, and we have chosen $L^{-1}\vec{p} = 0$ in the last line. Substituting this into Eq. (1.6) gives

$$\langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip \cdot (x_1 - x_2)} |\langle 0|\hat{\phi}(0)|\vec{p}=0\rangle|^2 + (1 \leftrightarrow 2) + \dots \quad (1.10)$$

where the omitted terms come from the contribution from states containing 2 or more particles. The constant matrix element

$$\boxed{Z \stackrel{\text{def}}{=} |\langle 0 | \hat{\phi}(0) | \vec{p} = 0 \rangle|^2} \quad (1.11)$$

will play an important role in higher order corrections, as we will see below. This gives the probability that the field operator $\hat{\phi}$ creates a single particle state. (Of course, this Z is completely different from the generating function of diagrams, also denoted by Z .) $Z = 1$ in free field theory, but $Z \neq 1$ at higher orders in the perturbative expansion.¹

We can write the momentum integral in Eq. (1.10) in a manifestly Lorentz invariant way using the identity

$$\int \frac{d\omega}{2\pi} \frac{ie^{-i\omega t}}{\omega^2 - E^2 + i\epsilon} = \frac{e^{-iE|t|}}{2E}. \quad (1.13)$$

This is a standard identity in quantum field theory that can be derived as follows. The integrand has poles at $\omega = \pm E \mp i\epsilon$. For $t > 0$, the ω integration contour can be completed in the lower half plane, while for $t < 0$, the integral can be completed in the upper half plane. The contribution from the residues gives the result above. This gives

$$\begin{aligned} \langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle &= Z \theta(x_1^0 - x_2^0) \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x_1 - x_2)} \frac{i}{p^2 - m_{\text{phys}}^2 + i\epsilon} \\ &\quad + (1 \leftrightarrow 2) + \dots \end{aligned} \quad (1.14)$$

We recognize

$$\Delta_{\text{phys}}(x_1, x_2) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x_1 - x_2)} \frac{1}{p^2 - m_{\text{phys}}^2 + i\epsilon} \quad (1.15)$$

as the Feynman propagator with the mass given by the physical mass. We therefore have

$$\boxed{\langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle = iZ \Delta_{\text{phys}}(x_1, x_2) + \dots} \quad (1.16)$$

¹In fact, an argument similar to the given above can be used to show that $Z < 1$ in any interacting theory. The argument proceeds by starting with the vacuum expectation value of the canonical commutation relation and inserting a complete set of states. In a free field theory this is saturated by the 1-particle states, and we get $Z = 1$. In an interacting theory, higher particle states contribute, and we get $Z < 1$. Because Z is manifestly positive, we conclude that

$$0 < Z < 1. \quad (1.12)$$

In momentum space, we see that the 2-point function has a pole at the physical mass. We have not discussed the contribution from states containing 2 or more particles, but it can be shown that they do not give rise to poles.

This result is important because it gives us a definition of the physical mass that is useful for calculations: the mass is the pole in the 2-point function. For this reason, the physical mass is sometimes referred to as the ‘pole mass.’

1.2 The 1PI 2-Point Function

Consider the momentum space 2-point function with external propagators removed (‘amputated’). This is given by a sum of diagrams. A given diagram is called ‘1 particle irreducible’ (1PI) if it cannot be cut into two disconnected pieces by cutting a single internal line. For example,

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \circ \end{array} \quad (1.17)$$

is 1PI, while

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \circ \text{---} \bullet \text{---} \\ \vdots \end{array} \quad (1.18)$$

is not, since it can be cut in half as shown. To emphasize that the external lines have been amputated, we have put a small slash on them. For example, this reminds us that we cannot cut a diagram in half by cutting an external line. We denote the sum of all 1PI 2-point graphs as

$$\boxed{\text{---} \textcircled{\text{1PI}} \text{---} \leftarrow p} = -i\Sigma(p^2). \quad (1.19)$$

For example, the diagrammatic expansion of Σ in ϕ^4 theory is

$$-i\Sigma = \begin{array}{c} \text{---} \bullet \text{---} \\ \circ \end{array} + \begin{array}{c} \text{---} \bullet \text{---} \\ \circ \text{---} \bullet \text{---} \\ \circ \end{array} + \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} + \mathcal{O}(\lambda^2). \quad (1.20)$$

We can express the full connected 2-point function (including external lines) in terms of the 1PI 2-point function as follows:

$$\text{---} \bullet \text{---} \leftarrow p = \text{---} \text{---} + \text{---} \textcircled{\text{1PI}} \text{---} + \text{---} \textcircled{\text{1PI}} \textcircled{\text{1PI}} \text{---} + \dots \quad (1.21)$$

$$\begin{aligned}
&= \frac{i}{p^2 - m^2} + \left(\frac{i}{p^2 - m^2} \right)^2 (-i\Sigma(p^2)) \\
&\quad + \left(\frac{i}{p^2 - m^2} \right)^3 (-i\Sigma(p^2))^2 + \dots
\end{aligned} \tag{1.22}$$

Factoring out one propagator, we see that this is a geometric series:

$$\begin{aligned}
\text{---} \circlearrowleft \text{---} \xleftarrow{p} &= \frac{i}{p^2 - m^2} \sum_{n=0}^{\infty} \left(\frac{\Sigma(p^2)}{p^2 - m^2} \right)^n \\
&= \frac{i}{p^2 - m^2} \left[1 - \frac{\Sigma(p^2)}{p^2 - m^2} \right]^{-1}.
\end{aligned} \tag{1.23}$$

Putting all terms over a common denominator gives

$$\boxed{\text{---} \circlearrowleft \text{---} \xleftarrow{p} = \frac{i}{p^2 - m^2 - \Sigma(p^2)}} \tag{1.24}$$

This result is sometimes called **Dyson's formula**.² It is an important result: it tells us that the 1PI corrections can be resummed to appear as a correction to the denominator of the full 2-point function.

1.3 Diagrammatic Expansion of the Physical Mass

To find the relation between Σ and the physical mass, we expand the denominator of the Dyson formula around $p^2 = m_{\text{phys}}^2$:

$$\begin{aligned}
p^2 - m^2 - \Sigma(p^2) &= p^2 - m^2 - \left[\Sigma(m_{\text{phys}}^2) + \Sigma'(m_{\text{phys}}^2)(p^2 - m_{\text{phys}}^2) \right. \\
&\quad \left. + \mathcal{O}((p^2 - m_{\text{phys}}^2)^2) \right] \\
&= \left[1 - \Sigma'(m_{\text{phys}}^2) \right] (p^2 - m_{\text{phys}}^2) + \mathcal{O}((p^2 - m_{\text{phys}}^2)^2).
\end{aligned} \tag{1.25}$$

The terms we have omitted do not contribute to the pole. Demanding that the pole is at $p^2 = m_{\text{phys}}^2$ requires

$$\boxed{m_{\text{phys}}^2 = m^2 + \Sigma(p^2 = m_{\text{phys}}^2)} \tag{1.26}$$

²Because the convergence of the series Eq. (1.23) may be in doubt, we should really take Eq. (1.24) as the definition of Σ . Reversing the argument above, we conclude that Σ is given by the sum of all 1PI diagrams.

Notice that for large k , we can approximate both of the propagators by $1/k^2$. The large k region of integration therefore behaves like

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4}. \quad (2.3)$$

This diverges for $d \geq 4$, so the expression Eq. (2.2) is ill-defined for these values of d . We will discuss how to deal with this situation in detail later. For now we want to avoid this complication, so we simply take $d = 3$.

We now introduce some tricks (also due to Feynman) for calculating loop integrals such as Eq. (2.2).

- *Combine denominators:* We use the identity

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} \quad (2.4)$$

to write the integral in terms of a single denominator. This is called a **Feynman parameter integral**. In the present case, we have

$$\begin{aligned} \text{denominator} &= x(k^2 + 2p \cdot k + p^2 - m^2 + i\epsilon) + (1-x)(k^2 - m^2 + i\epsilon) \\ &= k^2 + 2xp \cdot k + xp^2 - m^2 + i\epsilon \\ &= K^2 - M^2 + i\epsilon, \end{aligned} \quad (2.5)$$

where

$$K = k + xp, \quad M^2 = m^2 - x(1-x)p^2. \quad (2.6)$$

- *Shift the momentum integral:* Writing the integral in terms of the shifted variable K , we have

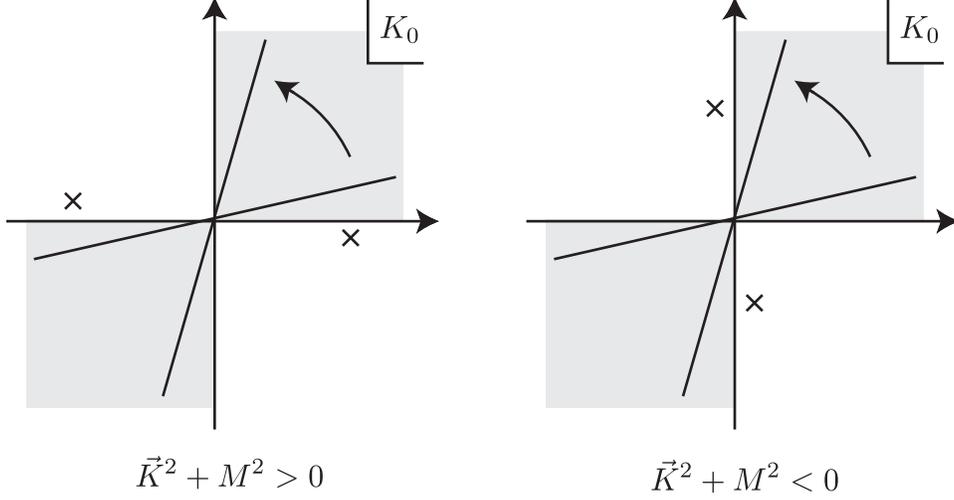
$$\Sigma(p^2) = \frac{i\lambda^2}{2} \int_0^1 dx \int \frac{d^3 K}{(2\pi)^3} \frac{1}{(K^2 - M^2 + i\epsilon)^2}. \quad (2.7)$$

- *Wick rotate:* The integrand has poles in the K_0 plane at

$$K_0 = \pm \sqrt{\vec{K}^2 + M^2 - i\epsilon}. \quad (2.8)$$

The position of the poles depends on the sign of $\vec{K}^2 + M^2$ (note that M^2 can be

negative):



For either sign, we can rotate the contour to imaginary K_0 :

$$K_0 = +iK_{E0}, \quad K_{E0} = \text{real}. \quad (2.9)$$

In terms of these ‘Euclidean’ momenta, we have

$$d^3K = id^3K_E, \quad K^2 = -K_E^2 = -(K_{E0}K_{E0} + K_{Ej}K_{Ej}), \quad (2.10)$$

where $j = 1, 2$. We can then write

$$\Sigma(p^2) = -\frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^3K_E}{(2\pi)^3} \frac{1}{(K_E^2 + M^2 - i\epsilon)^2}. \quad (2.11)$$

It is important to note that the external momentum p has not been Wick rotated, and is still the physical momentum.

- *Perform momentum integral:* The integrand is spherically symmetric in Euclidean momentum space, so we can write

$$d^3K_E = 4\pi K_E^2 dK_E, \quad (2.12)$$

where K_E now denotes the length of the Euclidean momentum. This gives an elementary integral

$$\begin{aligned} \Sigma(p^2) &= -\frac{\lambda^2}{4\pi^2} \int_0^1 dx \int_0^\infty dK_E \frac{K_E^2}{(K_E^2 + M^2 - i\epsilon)^2} \\ &= -\frac{\lambda^2}{4\pi^2} \int_0^1 dx \frac{\pi}{4\sqrt{M^2 - i\epsilon}}. \end{aligned} \quad (2.13)$$

We have obtained the loop integral in terms of a Feynman parameter integral:

$$\Sigma(p^2) = -\frac{\lambda^2}{16\pi} \int_0^1 dx [m^2 - x(1-x)p^2 - i\epsilon]^{-1/2}. \quad (2.14)$$

These steps (combining denominators, Wick rotating, and performing the momentum integrals) can be straightforwardly carried out for any diagram. Performing the Feynman parameter integrals is difficult in general, although we can do it for this simple example.

The $i\epsilon$ factor is important for resolving the poles if the term in square brackets is negative. Using the fact that $\max[x(1-x)] = \frac{1}{4}$, we see that this occurs only if

$$p^2 \geq 4m^2. \quad (2.15)$$

This is precisely the condition that p^2 is large enough to allow the production of a pair of particles of mass m . (Note that p^2 is the square of the center of mass energy.) That is, if $p^2 \geq 4m^2$, the momenta of the intermediate lines can satisfy $k^2 = m^2$, $(k+p)^2 = m^2$; in this case, we say that the momenta in the loop are **on shell**. As long as $p^2 < 4m^2$ we can ignore the $i\epsilon$ factor, and the integrand is manifestly real and analytic in p^2 . In particular, the fact that it is real near $p^2 = m^2$ is crucial in order for us to interpret it as a correction to the physical mass. For $p^2 \geq 4m^2$, singularities can (and do) appear. This is a particular case of an important general result: loop diagrams have singularities in momentum space in regions where the momenta of the intermediate lines can go on shell.

It may appear that the connection between the singularities of the correlation functions and the existence of real intermediate states is violated by the fact that m^2 is not the physical mass. However, we will see that it is possible to reorganize the perturbative expansion so that the propagator used in perturbation theory has poles at $p^2 = m_{\text{phys}}^2$. We will show below that in this ‘renormalized’ expansion, singularities of correlation functions are always due to real intermediate states.

To evaluate the integral Eq. (2.14), we scale out the m dependence by writing

$$\Sigma(p^2) = -\frac{\lambda^2}{16\pi m} \int_0^1 dx [1 - x(1-x)z]^{-1/2}, \quad (2.16)$$

where

$$z = \frac{p^2 + i\epsilon}{m^2}. \quad (2.17)$$

Performing the integral, we obtain

$$\Sigma(p^2) = -\frac{\lambda^2}{16\pi} \frac{1}{\sqrt{p^2 + i\epsilon}} \ln \frac{1 + \sqrt{p^2/4m^2 + i\epsilon}}{1 - \sqrt{p^2/4m^2 + i\epsilon}}. \quad (2.18)$$

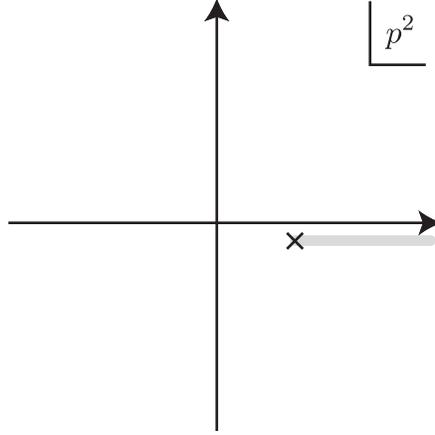
This function is not singular at $p^2 = 0$ because the singularity in the prefactor is cancelled by the zero of the logarithm:

$$\Sigma(p^2) = -\frac{\lambda^2}{16\pi m} \left[1 + \frac{p^2}{6m^2} + \mathcal{O}(p^4) \right]. \quad (2.19)$$

There is a branch cut singularity when $2m + \sqrt{p^2 + i\epsilon}$ goes to zero, which occurs at

$$p^2 = 4m^2 - i\epsilon. \quad (2.20)$$

In the complex p^2 plane, the singularity structure looks like



The $i\epsilon$ tells us that the singularity should be approached from the direction of positive imaginary p^2 . In this example, there are no singularities in the entire complex p^2 plane other than those from real intermediate states. Later, we will prove that this result is general if we restrict to real external momenta.

We complete our treatment of this example by computing m_{phys} and Z . To compute m_{phys} , we need to find the position of the pole in the full 2-point function. We must therefore solve

$$p^2 - m^2 - \Sigma(p^2) \Big|_{p^2 = m_{\text{phys}}^2} = 0. \quad (2.21)$$

Since $m_{\text{phys}}^2 = m^2 + \mathcal{O}(\lambda^2)$, we can write

$$m_{\text{phys}}^2 = m^2 + \Sigma(m^2) + \mathcal{O}(\lambda^4). \quad (2.22)$$

To evaluate $\Sigma(m^2)$, we can ignore the $i\epsilon$ factor and obtain

$$\begin{aligned} \Sigma(m^2) &= -\frac{\lambda^2}{16\pi m} \int_0^1 dx [1 - x(1-x)]^{-1/2} \\ &= -\frac{\lambda^2}{16\pi m} \ln 3. \end{aligned} \quad (2.23)$$

We see that the loop corrections make the physical mass smaller than m^2 .

We can also compute Z using Eq. (1.29). To the order we are working, we have

$$Z = 1 + \Sigma'(m^2) + \mathcal{O}(\lambda^4), \quad (2.24)$$

where $\Sigma'(p^2) = \partial\Sigma(p^2)/\partial p^2$. Using Eq. (2.14), we have

$$\Sigma'(p^2) = -\frac{\lambda}{32\pi} \int_0^1 dx x(1-x) [m^2 - x(1-x)p^2 - i\epsilon]^{-3/2}. \quad (2.25)$$

Therefore,

$$\Sigma'(m^2) = -\frac{\lambda^2}{32\pi m^3} \int_0^1 dx x(1-x) [1 - x(1-x)]^{-3/2} \quad (2.26)$$

$$= -\frac{\lambda^2}{32\pi m^3} \left(\frac{4}{3} - \ln 3 \right). \quad (2.27)$$

Note that $\frac{4}{3} - \ln 3 \simeq 0.23 > 0$, so $Z < 1$, as expected. (In fact, the Feynman integral in Eq. (2.26) is manifestly positive.)

2.1 Mass Renormalization

For some purposes, it is convenient to define the diagrammatic expansion so that the mass in the propagator is the same as the physical mass to all orders in perturbation theory. To do this, we write

$$m^2 = m_{\text{phys}}^2 + \Delta m^2 \quad (2.28)$$

and note that Δm^2 starts at higher orders in the loop expansion. Here, m^2 is the mass parameter that appears in the Lagrangian, sometimes called the **bare mass**.

As discussed previously, the loop expansion can be viewed as an expansion in powers of \hbar , so $\Delta m^2 = \mathcal{O}(\hbar)$ is a quantum correction. We then expand systematically in powers of \hbar , treating m_{phys}^2 as $\mathcal{O}(\hbar^0)$. We therefore write the Lagrangian as

$$\mathcal{L} = \mathcal{L}_{\text{ren}} + \Delta\mathcal{L}, \quad (2.29)$$

where

$$\mathcal{L}_{\text{ren}} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m_{\text{phys}}^2 \phi^2 + \frac{\lambda}{3!} \phi^3 \quad (2.30)$$

is the ‘renormalized’ Lagrangian and

$$\Delta\mathcal{L} = -\frac{1}{2} \Delta m^2 \phi^2 \quad (2.31)$$

is a ‘counterterm’ Lagrangian. Because $\Delta\mathcal{L}$ is treated as an $\mathcal{O}(\hbar)$ perturbation, it gives rise to a new Feynman rule:

$$\text{---}\blacksquare\text{---} = -i\Delta m^2. \quad (2.32)$$

This is to be regarded as a two-point vertex. The square dot on the vertex reminds us that it is a counterterm. The expansion of the 1PI 2-point function to $\mathcal{O}(\hbar)$ is therefore

$$-i\Sigma = \text{---}\bullet\text{---} + \text{---}\blacksquare\text{---} + \mathcal{O}(\hbar^2), \quad (2.33)$$

where the counterterm Δm^2 is determined by the condition that m_{phys} is the physical mass. In equations, this is

$$\Sigma(p^2 = m_{\text{phys}}^2) = 0. \quad (2.34)$$

By imposing this **renormalization condition** order by order in the \hbar expansion, we ensure that m_{phys} is the physical mass at each order in perturbation theory. In this perturbative expansion, the Feynman propagator is

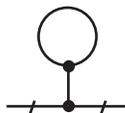
$$\Delta(x_1, x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x_1-x_2)}}{p^2 - m_{\text{phys}}^2 + i\epsilon} \quad (2.35)$$

to all orders in perturbation theory. The original theory was defined by two parameters, λ and m^2 . In the ‘renormalized’ perturbative expansion, we have traded the parameter m^2 for m_{phys}^2 . The condition Eq. (2.35) determines Δm^2 order by order in the expansion, so it is not an independent parameter.

In this way of organizing the perturbative expansion, the position of the poles in the propagators coincide with the physical poles. This is convenient for proving unitarity from the diagrammatic expansion, as we will discuss below.

2.2 Tadpole Condition

There is another diagram that contributes to the 2-point function at $\mathcal{O}(\lambda^2)$, namely



(Although this diagram can be cut into two parts by cutting the internal line, it contributes to Σ because of Eq. (1.21).) Associated with this is a contribution to the 1-point function

$$\text{---}\bullet\text{---} = -\frac{i\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{i}{k^2 - m^2 + i\epsilon} \quad (2.36)$$

A diagram like this that contributes to the 1-point function is called a **tadpole diagram** for obvious reasons. This integral is ill-defined because it diverges at large k , and therefore the contribution to the 2-point function is also ill-defined.

To understand what is going on, note that we can get a tree-level contribution to the 1-point function by adding a linear term in ϕ to the Lagrangian:

$$\Delta\mathcal{L} = -\kappa\phi. \quad (2.37)$$

This term is allowed by all symmetries, and there is no reason not to add it. (Since the interaction already contains a ϕ^3 term, we cannot forbid this by imposing $\phi \mapsto -\phi$ symmetry.) This term would give rise to a 1-point vertex in the Feynman rules:

$$\begin{array}{c} \bullet \\ | \end{array} = -i\kappa \quad (2.38)$$

This would also contribute to the 1-point function. The significance of this term is clearer if we consider the full potential:

$$V(\phi) = \kappa\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{3!}\phi^3. \quad (2.39)$$

This potential is minimized at $\phi = v$, where

$$v = -\frac{m^2 - \sqrt{m^4 - 2\lambda\kappa}}{\lambda}, \quad (2.40)$$

assuming that $m^4 \geq 2\lambda\kappa$. (If $m^4 < 2\lambda\kappa$ the potential has no stationary points.) Expanding around $\phi = v$,

$$\phi = v + \phi', \quad (2.41)$$

the potential is

$$V(\phi') = \frac{1}{2}m'^2\phi'^2 + \frac{\lambda}{3!}\phi'^3, \quad (2.42)$$

where $m'^2 = m^2 + \lambda v$. We see that expanding about $\phi = v$ gives a theory with no linear term in the potential. The value of κ can be absorbed into a shift in the field.

We see that demanding that the potential be extremized is equivalent to demanding the absence of a tadpole at tree level. It is therefore natural to impose the **tadpole condition**

3 Analyticity of Feynman Amplitudes

We address the question of the location of the singularities of Feynman diagrams such as the example above. By a ‘singularity,’ we mean any point where the amplitude is not analytic as a function of external momenta. We therefore consider a branch point as a singularity, even though the amplitude does not blow up there. We will show here that *all* singularities of Feynman diagrams are associated with the appearance of real intermediate states. We might expect that a Feynman diagram has a singularity whenever any one of the internal propagators can go on-shell. However, in most cases we can deform the integral in the complex plane to avoid the singularity. A singularity can occur only if the integrand has a singularity that cannot be avoided.

Let us see how this works for integrals of the form

$$I(x) = \int_a^b dz \frac{N(x, z)}{D(x, z)}, \quad (3.1)$$

where N and D are analytic in a neighborhood that includes the real interval $a \leq z \leq b$ and a region around the x values of interest. An analytic function can be written as a convergent power series, and therefore the integrand is singular only at zeros of D . (Note that the Feynman parameter integral in Eq. (2.14) is *not* of this form because the denominator has a square-root singularity in the physical region.) Suppose that for $D(\tilde{x}, \tilde{z}) = 0$ for some $a < \tilde{z} < b$, but $\partial_z D(\tilde{x}, \tilde{z}) \neq 0$. Then we can deform the contour in the complex z plane to avoid the singularity, and $I(x)$ is nonsingular at $x = \tilde{x}$. The only singularities of $I(x)$ result from singularities of the integrand that cannot be avoided by deforming the z contour of integration. This happens in one of the following two cases:

- Singularities at the endpoints, *i.e.* $D(\tilde{x}, a) = 0$ or $D(\tilde{x}, b) = 0$.
- “Pinch” singularities where a zero approaches the the real z axis from both sides as $x \rightarrow \tilde{x}$. This occurs if D has a double zero in z :

$$D(\tilde{x}, \tilde{z}) = 0 \quad \text{and} \quad \partial_z D(\tilde{x}, \tilde{z}) = 0. \quad (3.2)$$

To see this, we expand D about $z = \tilde{z}$, $x = \tilde{x}$:

$$D(x, z) = A(z - \tilde{z})^2 + B(x - \tilde{x})^n + \dots \quad (3.3)$$

for some A and B and integer $n \geq 1$. In the approximation where we neglect higher order terms, D has zeros at

$$z = \tilde{z} \pm \sqrt{\frac{B}{A}}(x - \tilde{x})^n. \quad (3.4)$$

To move the singularity off the real z axis, we must choose the phase of $x - \tilde{x}$ so that the square root has a nonzero imaginary part. But then there are two zeros that “pinch” the real z axis from both sides as $x \rightarrow \tilde{x}$.

Some simple examples:

$$I(x) = \int_0^1 \frac{dz}{z+x} = \ln \frac{1+x}{x}. \quad (3.5)$$

The denominator has a simple zero when $z+x=0$, but this is an avoidable singularity unless it is at an endpoint, that is $x=0$ or $x=-1$. We see that this does account for the singular points of the integral. Note that the singularity is a branch cut singularity. We can move the branch cut by using a different branch of the logarithm, but we cannot move the branch points. Another example is

$$I(x) = \int_{-\infty}^{\infty} \frac{dz}{z^2+x} = \frac{\pi}{\sqrt{x}}. \quad (3.6)$$

The integrand has a pinch singularity when $x \rightarrow 0$, and indeed it is singular there.

We can extend this to multiple integrals of the form

$$I(x_1, \dots, x_m) = \int_{a_1}^{b_1} dz_1 \cdots \int_{a_n}^{b_n} dz_n \frac{N(x_1, \dots, x_m, z_1, \dots, z_n)}{D(x_1, \dots, x_m, z_1, \dots, z_n)}, \quad (3.7)$$

where N and D are analytic in a neighborhood that includes the real intervals $a_i \leq z_i \leq b_i$ for $i = 1, \dots, n$ and the values of x_1, \dots, x_m that are of interest. In this case, we must have either a pinch or endpoint singularity in each integration variable z_i . The condition for a singularity is that there are points $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{z}_1, \dots, \tilde{z}_n$, with $a_i \leq \tilde{z}_i \leq b_i$ such that $D(\tilde{x}, \tilde{z}) = 0$ and for each \tilde{z}_i we have either an endpoint singularity ($\tilde{z}_i = a_i$ or b_i) or a pinch singularity: $\partial_{z_i} D(\tilde{x}, \tilde{z}) = 0$.

For example, consider the integral

$$I(x) = \int_{-1}^1 dz_1 \int_{-\infty}^{\infty} dz_2 \frac{1}{z_1 + z_2^2 + x} = 2\pi \left(\sqrt{1+x} - \sqrt{1-x} \right). \quad (3.8)$$

As $x \rightarrow 0$, the denominator of the integrand vanishes for $z_1 = z_2 = 0$. It is a pinch singularity in z_2 , but it is avoidable in z_1 . Therefore, there is no singularity of $I(x)$ at $x = 0$. For $x = \pm 1$, the denominator vanishes for $z_1 = \pm 1, z_2 = 0$. This is an endpoint singularity in z_1 and a pinch singularity in z_2 , so we expect a singularity there. The conditions for the singularity are therefore

Let us now apply this to a general Feynman diagram. A diagram with E external legs, I internal legs, and L loops can be written in the form

$$\begin{aligned} \mathcal{M}(p_1, \dots, p_E) &= \int d^4 k_1 \cdots d^4 k_L N(p_1, \dots, p_E, k_1, \dots, k_L) \\ &\times \frac{1}{q_1^2 - m_1^2 + i\epsilon} \cdots \frac{1}{q_I^2 - m_I^2 + i\epsilon}. \end{aligned} \quad (3.9)$$

Here p_1, \dots, p_E are external momenta, k_1, \dots, k_L are the loop momenta, and q_1, \dots, q_I are the momenta on the internal lines. The function N is a polynomial in the momenta arising from possible derivative couplings. (We assume that the amplitude is amputated, so there are no poles associated with external lines.) The internal momenta are linear combinations of the external momenta and the loop momenta that depend on the topology of the diagram. We can combine the denominators using the Feynman identity

$$\frac{1}{D_1 \cdots D_I} = \int_0^1 dx_1 \cdots \int_0^1 dx_I \delta(x_1 + \cdots + x_I - 1) \frac{1}{D}, \quad (3.10)$$

where

$$D = x_1 D_1 + \cdots + x_I D_I. \quad (3.11)$$

We therefore obtain

$$\mathcal{M}(p) = \int d^4 k_1 \cdots d^4 k_L \int_0^1 dx_1 \cdots \int_0^1 dx_I \delta(x_1 + \cdots + x_I - 1) \frac{N(p, k)}{[D(x, q)]^I}, \quad (3.12)$$

where

$$D(x, q) = x_1(q_1^2 - m_1^2 + i\epsilon) + \cdots + x_I(q_I^2 - m_I^2 + i\epsilon). \quad (3.13)$$

For purposes of discussing the analyticity structure of the diagram, the delta function is not very convenient. We can eliminate it using the following trick. First, note that there is no harm in extending the range of the x integrals from 0 to ∞ , since the delta function determines the range in any case. We then change variables $x_i \rightarrow x_i/\lambda$ where λ is a positive constant. This gives

$$\mathcal{M}(p) = \lambda \int d^4 k_1 \cdots d^4 k_L \int_0^\infty dx_1 \cdots \int_0^\infty dx_I \delta(x_1 + \cdots + x_I - \lambda) \frac{N(p, k)}{[D(x, q)]^I}, \quad (3.14)$$

where the factor of λ arises from the delta function. Since \mathcal{M} is independent of λ , we can write

$$\mathcal{M}(p) = \int_0^\infty d\lambda e^{-\lambda} \mathcal{M}(p) \quad (3.15)$$

$$\begin{aligned} &= \int_0^\infty d\lambda, e^{-\lambda} \lambda \int d^4 k_1 \cdots d^4 k_L \int_0^\infty dx_1 \cdots \int_0^\infty dx_I \\ &\quad \times \delta(x_1 + \cdots + x_I - \lambda) \frac{N(p, k)}{[D(x, q)]^I} \\ &= \int d^4 k_1 \cdots d^4 k_L \int_0^\infty dx_1 \cdots \int_0^\infty dx_I \frac{X e^{-X} N(p, k)}{[D(x, q)]^I}, \end{aligned} \quad (3.16)$$

where

$$X = x_1 + \cdots + x_I. \quad (3.17)$$

The integrand in Eq. (3.16) has singularities only when the denominator vanishes, and the denominator is a polynomial in the integration variables, so we can directly apply the criteria above to find the singularities.

In order to have a singularity, the integral must have an endpoint or pinch singularity in each of the $4L + I$ integrals. This means that for some value of the external momenta \tilde{p} and integration variables \tilde{k} and \tilde{x} we have $D(\tilde{x}, \tilde{q}) = 0$, such that the following conditions are satisfied.

- For each Feynman parameter integral over x_i ($i = 1, \dots, I$), either

$$\boxed{\tilde{x}_i = 0} \quad (3.18)$$

or

$$0 = \frac{\partial}{\partial x_i} D(\tilde{x}, \tilde{q}) = \tilde{q}_i^2 - m^2, \quad (3.19)$$

i.e.

$$\boxed{\tilde{q}_i^2 = m_i^2}. \quad (3.20)$$

Note that if Eq. (3.18) or Eq. (3.20) is satisfied for all i , then $D(\tilde{x}, \tilde{q}) = 0$, so we need not impose this condition independently.

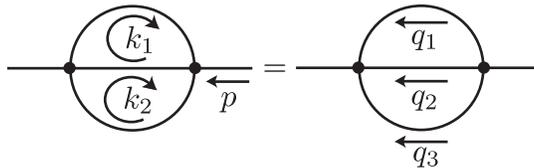
- For each loop integral over k_ℓ ($\ell = 1, \dots, L$) we must have a pinch singularity, since there are no endpoints to the integral. This means that

$$0 = \frac{\partial}{\partial k_{\ell\mu}} D(\tilde{x}, \tilde{q}) = 2 \sum_{i=1}^I \tilde{x}_i q_i^\nu \frac{\partial q_{i\nu}}{\partial k_{\ell\mu}} \Big|_{q=\tilde{q}}. \quad (3.21)$$

The partial derivatives satisfy

$$\frac{\partial q_{i\nu}}{\partial k_{\ell\mu}} = C_{i\ell} \delta_\nu^\mu, \quad (3.22)$$

where $C_{i\ell} = 0$ or ± 1 . This is because each loop momentum goes through a given internal line at most once, and can go in one of two directions. For example, in the 2-loop diagram



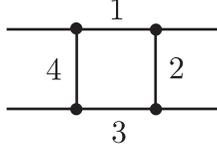
we have $q_1 = p - k_1$, $q_2 = k_1 - k_2$, $q_3 = k_2$. We can therefore restrict the sum Eq. (3.21) to the internal lines i with $C_{i\ell} \neq 0$. The coefficients $C_{i\ell} = \pm 1$ then just tell us whether \tilde{q}_i is going in the same direction or the opposite direction of the loop, and we can write the condition as

$$\boxed{\sum_{\text{any loop}} \tilde{x}_i \tilde{q}_{i\mu} = 0.} \quad (3.23)$$

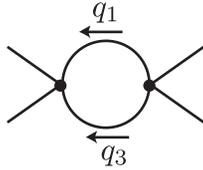
In the 2-loop example above, we have two loop conditions:

$$\begin{aligned} \tilde{x}_1 \tilde{q}_1 - \tilde{x}_2 \tilde{q}_2 &= 0, \\ \tilde{x}_2 \tilde{q}_2 - \tilde{x}_3 \tilde{q}_3 &= 0. \end{aligned} \quad (3.24)$$

The conditions for a singularity are therefore Eqs. (3.18) or (3.20) for each internal line, and Eq. (3.23) for each loop. These have a very simple physical interpretation. To understand it, it is useful to introduce a diagrammatic notation for the singularities in which a line with $\tilde{x}_i = 0$ is shrunk to zero. (We call this a “short circuited line.”) For example, in the diagram



we can short-circuit lines 2 and 4 to give a simpler diagram



The conditions for a singularity are therefore

$$\tilde{q}_1^2 = m_1^2, \quad \tilde{q}_3^2 = m_3^2, \quad (3.25)$$

and also

$$\tilde{x}_1 \tilde{q}_1 - \tilde{x}_3 \tilde{q}_3 = 0. \quad (3.26)$$

This means that \tilde{q}_1 and \tilde{q}_3 are parallel 4-vectors. (Note that $x_1, x_3 > 0$. If either Feynman parameter vanished, we would have a short-circuited line.) This implies that $\tilde{x}_1^2 m_1^2 = \tilde{x}_3^2 m_3^2$, or (since all quantities involved are positive)

$$\tilde{x}_1 m_1 = \tilde{x}_3 m_3. \quad (3.27)$$

The location of the singularity is therefore

$$s = (\tilde{q}_1 + \tilde{q}_3)^2 = (m_1 + m_3)^2. \quad (3.28)$$

Therefore, the condition for the appearance of a singularity is that there is enough energy to produce the particles 1 and 3.

Note that for any diagram, there is a trivial solution where all the internal lines are short-circuited, *i.e.* $\tilde{x}_1 = \dots = \tilde{x}_I = 0$. We can show that this does not give rise to a singularity by modifying Eq. (3.15) to read

$$\mathcal{M}(p) = e^\epsilon \int_\epsilon^\infty d\lambda e^{-\lambda} \mathcal{M}(p) \quad (3.29)$$

$$= e^\epsilon \int d^4 k_1 \dots d^4 k_L \int_0^\infty dx_1 \dots \int_0^\infty dx_I \theta(X - \epsilon) \frac{X e^{-X} N(k, p)}{[D(x, q)]^l}. \quad (3.30)$$

The left-hand side is independent of ϵ , but for any $\epsilon > 0$, the integrand vanishes when $x_1 = \dots = x_I = 0$. We conclude that at least one of the Feynman parameters must be nonzero in order to have a singularity. (The θ function does not eliminate the singularities where some of the \tilde{x}_i are nonzero, since the conditions Eq. (3.23) that determine the \tilde{x}_i do not determine the overall scale.)

So far, the conditions apply even for complex external momenta. For real external momenta, we can show that *all* singularities are associated with the production of particles in the intermediate states. In fact, the conditions above imply that the graph can be read as a picture of classical particles moving in spacetime, with all momenta real and on mass shell, and all particles moving forward in time. This is the **Coleman-Norton theorem**. To prove it, associate with each internal line a spacetime separation

$$\Delta_i^\mu = \tilde{x}_i \tilde{q}_i^\mu. \quad (3.31)$$

Since all the \tilde{x}_i and \tilde{q}_i are real, this is real. It also satisfies the consistency condition

$$\sum_{\text{any loop}} \Delta_i^\mu = 0. \quad (3.32)$$

For any non-short circuited line, the particles are on mass shell. For a short-circuited line, $\tilde{x}_i = 0$ implies $\Delta_i^\mu = 0$, so the particle exists for zero time. (We can think of this as a classically allowed interaction mediated by the particle.) To see that the particles propagate forward in time, note that the spacetime interval associated with a classical particle with momentum \tilde{q}_i is

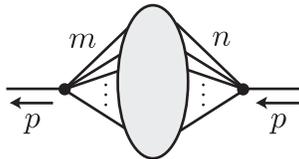
$$\Delta_i^\mu = \tau_i \frac{\tilde{q}_i^\mu}{m_i}, \quad (3.33)$$

where τ_i is the proper time. Comparing with the definition Eq. (3.31), we see that

$$\tau_i = \tilde{x}_i m_i > 0, \quad (3.34)$$

so the particle does indeed move forward in time.

Let us use these rules to analyze the singularities of the 1PI 2-point function $\Sigma(p^2)$. The graphs that contribute have the form



In this diagram, the vertex on the left connects the external line to m internal lines, and the vertex on the right connects to n internal lines. To be continued...

4 LSZ Reduction

We now show how to use these ideas to relate correlation functions to S matrix elements. Let us recall some basic ideas of scattering theory. A physical scattering experiment involves initial and final states consisting of well-separated wavepackets. Because the initial and final state wavepackets are separated, we can neglect interactions between them at sufficiently late and early times. The initial and final states can therefore be treated as energy eigenstates of a free theory; we call these ‘in’ states and ‘out’ states. Since we are working in Heisenberg picture, the states are independent of time and the fields evolve. Therefore, the information about the initial and final states is contained in free field operators $\hat{\phi}_{\text{in}}$ and $\hat{\phi}_{\text{out}}$. The S matrix is relation between the in and out fields:

$$\hat{\phi}_{\text{out}} = \hat{S} \hat{\phi}_{\text{in}} \hat{S}^\dagger. \quad (4.1)$$

As we have seen above, the interacting Heisenberg field operators $\hat{\phi}$ do not create 1-particle states with unit probability as free fields do. The probability to create a single particle state is given by the Z factor of Eq. (1.11), so we have

$$\hat{\phi}(x) \xrightarrow{x^0 \rightarrow -\infty} \sqrt{Z} \hat{\phi}_{\text{in}}(x), \quad \hat{\phi}(x) \xrightarrow{x^0 \rightarrow +\infty} \sqrt{Z} \hat{\phi}_{\text{out}}(x). \quad (4.2)$$

Here $\hat{\phi}_{\text{in}}$ and $\hat{\phi}_{\text{out}}$ are free fields acting on an ‘in’ and ‘out’ Fock space, respectively.

Now consider the momentum-space correlation function

$$\begin{aligned} & (2\pi)^4 \delta^4(p_1 + \dots + p_n) \tilde{G}^{(n)}(p_1, \dots, p_n) \\ &= \int d^4x_1 \dots \int d^4x_n e^{ip_1 \cdot x_1} \dots e^{ip_n \cdot x_n} \langle 0 | T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) | 0 \rangle. \end{aligned} \quad (4.3)$$

Using the same type of reasoning used for the 2-point function, we will show that this correlation function has poles corresponding to matrix elements of 1-particle states.

To see this, consider the region of integration where $x_1^0 \gg x_2^0, \dots, x_n^0$. In this regime, we can replace

$$\begin{aligned} \langle 0|T\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle &\rightarrow \langle 0|\sqrt{Z}\hat{\phi}_{\text{out}}(x_1)T\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{Z}}{2E_{\vec{k}}} e^{-ik\cdot x_1} {}_{\text{out}}\langle \vec{k}|T\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)|0\rangle + \cdots, \end{aligned} \quad (4.4)$$

where the omitted terms come from states with 2 or more particles. Substituting this into Eq. (4.3), the x_1^0 integral is

$$\int_T^{+\infty} dx_1^0 e^{ip_{1,0}x_1^0} e^{-iE_{\vec{k}}x_1^0} = \frac{i}{p_{1,0} - E_{\vec{k}}} e^{i(p_{1,0} - E_{\vec{k}})T}, \quad (4.5)$$

where $T = \max\{x_2^0, \dots, x_n^0\}$. The \vec{x}_1 integral is

$$\int d^3x_1 e^{-i\vec{p}_1\cdot\vec{x}_1} e^{i\vec{k}\cdot\vec{x}_1} = (2\pi)^3 \delta^3(\vec{k} - \vec{p}_1). \quad (4.6)$$

We therefore obtain a contribution to the momentum-space correlation function

$$\begin{aligned} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \tilde{G}^{(n)}(p_1, \dots, p_n) \\ = \frac{\sqrt{Z}}{2E_{\vec{p}_1}} \frac{i}{p_{1,0} - E_{\vec{p}_1}} \int d^4x_2 \cdots \int d^4x_n e^{ip_2\cdot x_2} \cdots e^{ip_n\cdot x_n} \\ \times {}_{\text{out}}\langle \vec{p}_1|T\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)|0\rangle + \cdots \end{aligned} \quad (4.7)$$

Note that near $p_{1,0} = E_{\vec{p}_1}$,

$$\frac{1}{2E_{\vec{p}_1}} \frac{i}{p_{1,0} - E_{\vec{p}_1}} = \frac{i}{p^2 - m_{\text{phys}}^2} + \text{non-pole}, \quad (4.8)$$

so we have identified a pole term in the momentum-space correlation function:

$$\begin{aligned} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \tilde{G}^{(n)}(p_1, \dots, p_n) \\ \xrightarrow{p_{1,0} \rightarrow +E_{\vec{p}_1}} \frac{i\sqrt{Z}}{p_1^2 - m_{\text{phys}}^2} \int d^4x_2 \cdots \int d^4x_n e^{ip_2\cdot x_2} \cdots e^{ip_n\cdot x_n} \\ \times {}_{\text{out}}\langle \vec{p}_1|T\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)|0\rangle. \end{aligned} \quad (4.9)$$

It can be shown that states containing 2 or more particles and other regions of integration do not give rise to a pole at $p_{1,0} \rightarrow +E_{\vec{p}_1}$.

Let us now consider the region of integration $x_1^0 \ll x_2^0, \dots, x_n^0$. In this regime, we can replace

$$\begin{aligned} \langle 0|T\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle &\rightarrow \langle 0|[T\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)]\sqrt{Z}\hat{\phi}_{\text{in}}(x_1)|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{Z}}{2E_{\vec{k}}} e^{+ik\cdot x_1} \langle 0|T\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)|\vec{k}\rangle_{\text{in}} + \cdots \end{aligned} \quad (4.10)$$

Repeating the steps above, we find that the momentum-space correlation function has a pole at $p_{1,0} \rightarrow -E_{\vec{p}_1}$:

$$\begin{aligned} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \tilde{G}^{(n)}(p_1, \dots, p_n) \\ \xrightarrow{p_{1,0} \rightarrow -E_{\vec{p}_1}} \frac{i\sqrt{Z}}{p_1^2 - m_{\text{phys}}^2} \int d^4x_2 \cdots \int d^4x_n e^{ip_2\cdot x_2} \cdots e^{ip_n\cdot x_n} \\ \times \langle 0|T\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)|\vec{p}_1\rangle_{\text{in}}. \end{aligned} \quad (4.11)$$

We see that poles at positive (negative) energy correspond to incoming (outgoing) states.

We can continue in this way, identifying poles in each of the momenta with incoming or outgoing states. We obtain

$$\boxed{\begin{aligned} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \tilde{G}^{(n)}(p_1, \dots, p_n) \\ \rightarrow \frac{i\sqrt{Z}}{p_1^2 - m_{\text{phys}}^2} \cdots \frac{i\sqrt{Z}}{p_n^2 - m_{\text{phys}}^2} \text{out}\langle \vec{p}_1, \dots, \vec{p}_r | \vec{p}_{r+1}, \dots, \vec{p}_n \rangle_{\text{in}} \end{aligned}} \quad (4.12)$$

in the limit

$$\begin{aligned} p_{1,0} &\rightarrow +E_{\vec{p}_1}, \dots, p_{r,0} \rightarrow +E_{\vec{p}_r} \\ p_{r+1,0} &\rightarrow -E_{\vec{p}_{r+1}}, \dots, p_{n,0} \rightarrow -E_{\vec{p}_n}. \end{aligned} \quad (4.13)$$

Eq. (4.12) is the **LSZ reduction formula**, due to Lehmann, Symanzyk, and Zimmermann.

The matrix element in Eq. (4.12) is precisely an S matrix element:

$$\text{out}\langle \vec{p}_1, \dots, \vec{p}_r | \vec{p}_{r+1}, \dots, \vec{p}_n \rangle_{\text{in}} = \langle \vec{p}_1, \dots, \vec{p}_r | \hat{S} | \vec{p}_{r+1}, \dots, \vec{p}_n \rangle. \quad (4.14)$$

It is conventional to write

$$\boxed{\hat{S} = 1 + i\hat{T}}, \quad (4.15)$$

and to define the **invariant amplitude** \mathcal{M} by

$$\boxed{(2\pi)^4 \delta^4(p_1 + \cdots + p_n) \mathcal{M} = \langle \vec{p}_1, \dots, \vec{p}_r | \hat{T} | \vec{p}_{r+1}, \dots, \vec{p}_n \rangle.} \quad (4.16)$$

In terms of diagrams, we know that the momentum-space correlation function is written in terms of amputated diagrams multiplied by propagator corrections:

$$\begin{aligned} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \tilde{G}^{(n)}(p_1, \dots, p_n) &= \text{Diagram: A central circle labeled 'Amp' is connected to three other circles. The top circle has an incoming arrow labeled p_1. The bottom-left circle has an incoming arrow labeled p_2. The bottom-right circle has an outgoing arrow labeled p_n. Ellipses between the bottom circles indicate more external lines.} \\ &= \frac{i}{p_1^2 - m^2 - \Sigma(p_1^2)} \cdots \frac{i}{p_n^2 - m^2 - \Sigma(p_n^2)} \\ &\quad \times (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \tilde{G}_{\text{amp}}^{(n)}(p_1, \dots, p_n) \quad (4.17) \\ &\rightarrow \frac{iZ}{p_1^2 - m_{\text{phys}}^2} \cdots \frac{iZ}{p_n^2 - m_{\text{phys}}^2} \\ &\quad \times (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \tilde{G}_{\text{amp}}^{(n)}(p_1, \dots, p_n), \quad (4.18) \end{aligned}$$

where we take the on-shell limit in the last line. Comparing this with Eq. (4.16), we see that up to the factors of Z , \mathcal{M} is just the amputated correlation function evaluated on shell:

$$\boxed{i\mathcal{M} = \left(\sqrt{Z}\right)^n \tilde{G}_{\text{amp}}^{(n)} \Big|_{\text{on shell}},} \quad (4.19)$$

where the sign of the 0 component of the 4-momenta determines whether the corresponding state is incoming or outgoing (see Eq. (4.12)). This is our final result for expressing S -matrix elements in terms of correlation functions.

The factors of \sqrt{Z} in Eq. (4.19) can be understood from the fact that the S -matrix should not depend on rescaling of the field variable in the path integral. Consider the change of variables in the path integral

$$\phi = c\phi', \quad (4.20)$$

where c is a constant. Since this is just a change of variables, physical quantities such as S matrix elements should not be affected. Both $\tilde{G}_{\text{amp}}^{(n)}$ and Z in Eq. (4.19) depend on c , but it is not hard to see that the dependence cancels.

Exercise: Check that the c dependence cancels in Eq. (4.19) using the Feynman rules to deduce how $\tilde{G}^{(n)}$ and Z depend on c .

5 Unitarity

Unitarity of the S matrix has important implications for the structure of correlation function, which we now work out. Unitarity of the S matrix means

$$1 = \hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S}. \quad (5.1)$$

In terms of the transition matrix defined by Eq. (4.15), this implies

$$\hat{T} - \hat{T}^\dagger = i\hat{T}\hat{T}^\dagger = i\hat{T}^\dagger\hat{T}. \quad (5.2)$$

It is convenient to use the notation

$$T_{fi} = \langle f|\hat{T}|i\rangle, \quad (5.3)$$

and write this Eq. (5.2) as

$$T_{fi} - T_{if}^* = i \sum_{\alpha} T_{f\alpha} T_{i\alpha}^* = \sum_{\alpha} T_{\alpha i} T_{\alpha f}^*. \quad (5.4)$$

Here α is a complete set of states. In this notation, the invariant amplitude defined in Eq. (4.16) is given by

$$T_{fi} = (2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}_{fi}, \quad (5.5)$$

and we have

$$\boxed{\begin{aligned} \mathcal{M}_{fi} - \mathcal{M}_{if}^* &= i \sum_{\alpha} (2\pi)^4 \delta^4(p_i - p_{\alpha}) \mathcal{M}_{f\alpha} \mathcal{M}_{i\alpha}^* \\ &= i \sum_{\alpha} (2\pi)^4 \delta^4(p_i - p_{\alpha}) \mathcal{M}_{\alpha i} \mathcal{M}_{\alpha f}^*. \end{aligned}} \quad (5.6)$$

This expresses unitarity of the S matrix in terms of the invariant amplitudes.

5.1 The Optical Theorem

Specializing to the case $i = f$ we obtain the **generalized optical theorem**

$$\text{Im}(\mathcal{M}_{ii}) = \frac{1}{2} \sum_{\alpha} (2\pi)^4 \delta^4(p_i - p_{\alpha}) |\mathcal{M}_{i\alpha}|^2. \quad (5.7)$$

We have used a notation appropriate for discrete states, which corresponds to having a system in finite volume. Taking the infinite volume limit, we have

$$\begin{aligned} 1 &= \sum_{\alpha} |\alpha\rangle\langle\alpha| \\ &\rightarrow |0\rangle\langle 0| + \sum_{n=1}^{\infty} \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_1} \cdots \frac{d^3k_n}{(2\pi)^3} \frac{1}{2E_n} |k_1, \dots, k_n\rangle\langle k_1, \dots, k_n|. \end{aligned} \quad (5.8)$$

This is to be compared with n -body final state phase space

$$d\Phi_n(k_1, \dots, k_n; p) = \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_1} \cdots \frac{d^3k_n}{(2\pi)^3} \frac{1}{2E_n} (2\pi)^4 \delta^4(k_1 + \cdots + k_n - p). \quad (5.9)$$

This shows that we can write the generalized optical theorem as

$$\boxed{\text{Im}(\mathcal{M}_{ii}) = \frac{1}{2} \sum_{n=2}^{\infty} \int d\Phi_n(k_1, \dots, k_n; p_i) |\mathcal{M}(i \rightarrow k_1, \dots, k_n)|^2.} \quad (5.10)$$

We see that the right-hand side of the optical theorem is a reaction rate, up to initial state phase space factors.

One important special case is where \mathcal{M} is the matrix element for $2 \rightarrow 2$ scattering. In that case, the rate is related to the total cross section for the initial state, and we have

$$\boxed{\text{Im}(\mathcal{M}_{ii}) = 2E_{\text{cm}} |\vec{p}_{\text{cm}}| \sigma_{\text{tot}}(i \rightarrow \text{any}).} \quad (5.11)$$

This relation is conventionally called the ‘optical theorem’ in non-relativistic quantum mechanics. We see that it holds even in relativistic quantum field theory, and that it follows from unitarity alone.

5.2 Unstable Particles

We now discuss a very important application of unitarity to unstable particles. If a particle is unstable, then it cannot appear as an asymptotic state in the S matrix, and strictly we should not even refer to it as a ‘particle.’ However, physically it clearly does make sense to scatter unstable particles, if their lifetime is longer than the time needed to do the experiment. (For example, neutrons have a lifetime of about 15 minutes, plenty of time to do a scattering experiment.) The wavefunction of a particle of mass m oscillates as $\sim e^{-imt}$ in the particle rest frame, so if the decay rate Γ satisfies $\Gamma \ll m$ then the wavefunction will oscillate many times before it decays, and it makes sense to treat it as an approximate energy eigenstate.

We have seen above that unitarity requires Feynman diagrams to have a nonzero imaginary part whenever real intermediate states can appear. Applying this to the self-energy Σ , we see that real intermediate states can appear precisely when the particle is unstable. We therefore expect that Σ will have an imaginary part when it is unstable. In fact, in the limit $\Gamma \ll m$, we claim that we can identify

$$\boxed{\text{Im} \Sigma(p^2 = m^2) = -m\Gamma.} \quad (5.12)$$

(Here m is the physical mass.) To see this, let us take Eq. (5.12) as the definition of Γ for the moment. Near $p^2 = m^2$, we then have

$$\text{---} \circlearrowleft \text{---} = \frac{i}{p^2 - m^2 + im\Gamma} + \dots \quad (5.13)$$

Note that this moves the pole in the propagator off the real axis, in the same manner as the $i\epsilon$ prescription. To understand the physical meaning of this, suppose that the propagator appears in the s channel of a scattering process, *e.g.*

$$\begin{aligned} \text{---} \circlearrowleft \text{---} &= \text{---} \bullet \text{---} \circlearrowleft \text{---} \bullet \text{---} + \dots \\ &\propto \frac{i}{s - m^2 + im\Gamma} + \dots \end{aligned} \quad (5.14)$$

Squaring the amplitude, we see that the cross section near $s = m^2$ it is proportional to

$$\sigma \propto \frac{1}{(s - m^2)^2 + m^2\Gamma^2}, \quad (5.15)$$

which is the Breit-Wigner form of a resonance with decay rate Γ .

This result can also be heuristically derived from the generalized optical theorem above. The quantity $-i\Sigma$ was defined above to be the sum of all 1PI diagrams, but we can also identify it as the amplitude for $1 \rightarrow 1$ ‘scattering.’ This identification gives

$$\mathcal{M}(p \rightarrow p) = -\Sigma(p^2). \quad (5.16)$$

Applying the generalized optical theorem to \mathcal{M} , we see that the rate appearing on the right-hand side of Eq. (5.10) is the total decay rate of the particle, and we have

$$\text{Im}(\mathcal{M}(p \rightarrow p)) = m\Gamma. \quad (5.17)$$

A rigorous justification of Eq. (5.12) can be given using the diagrammatic version of the unitarity relation, described below.

5.3 Cutting Rules

The arguments above show that Eq. (5.6) is equivalent to the unitarity of the S matrix. We previously gave a formal argument that unitarity is automatic for the

Feynman rules obtained from the path integral. This means that we should be able to prove Eq. (5.6) directly from the Feynman rules obtained from the path integral. This gives a more rigorous justification for the claim that unitarity is automatic in the path integral approach. This argument can be carried out. The singularities of Feynman diagrams can be fully analyzed, and the imaginary parts of diagrams can be shown to satisfy Eq. (5.6) to all orders in perturbation theory. These are called the **cutting rules**, since they involve ‘cuts’ of diagrams corresponding to real intermediate states.

The original result is due to Cutkosky. There is a very elegant (but very terse) derivation of these rules in 't Hooft and Veltman's *Diagrammar*. I will cover this if time allows.