Correlation Functions and Diagrams

Correlation function of fields are the natural objects to study in the path integral formulation. They contain the physical information we are interested in (*e.g.* scattering amplitudes) and have a simple expansion in terms of Feynman diagrams. This chapter develops this formalism, which will be the language used for the rest of the course.

1 Sources

The path integral gives us the time evolution operator, which in principle contains all the information about the dynamics of the system. However, in order to use the path integral to do physics we need to find a way to describe initial and final particle states in path integral language. The way to do this is to couple the fields to spacetime-dependent background fields ("sources") that can create or destroy particles. For example, in our scalar field theory, we add a source field $J(x)$ coupled linearly to ϕ:

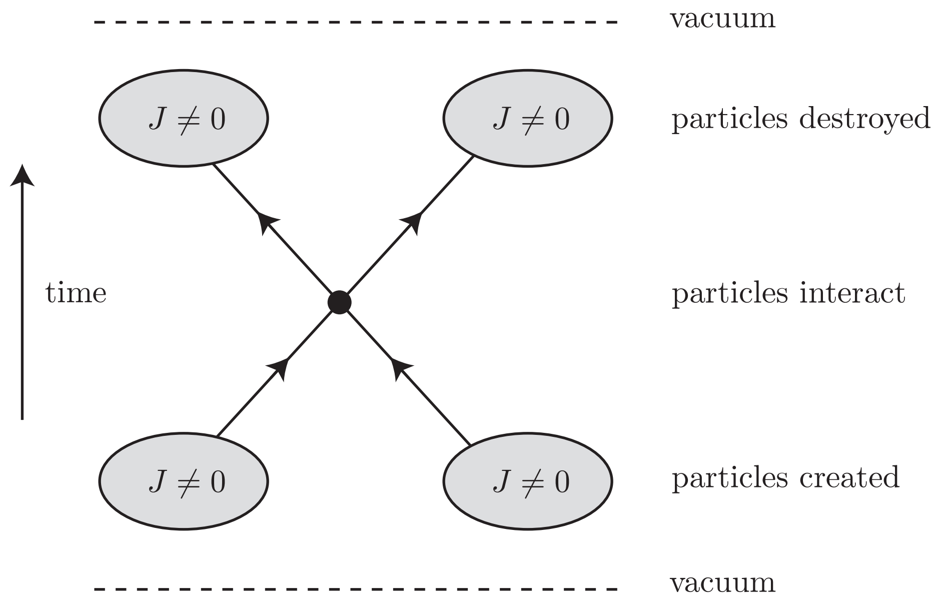
$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) + J\phi. \tag{1.1}$$

The source field $J(x)$ is not dynamical; it is a c-number field that acts as a background to the ϕ dynamics. (In path integral language, we integrate over configurations of ϕ, for a given J.) With the addition of the source term the classical equations of motion for ϕ are

$$\Box\phi + V'(\phi) = J. \tag{1.2}$$

We see that J plays the same role as an electromagnetic current in Maxwell's equations, which is why we call it a source.

Consider a source field that turns on briefly at some initial time, and is cleverly chosen so that it creates two particles with close to unit probability. This part of the field configuration therefore represents the initial state of the system. At a later time, we choose the background field to turn on briefly in a clever way so that it precisely absorbs two particles with a given configuration with close to unit probability. This part of the field configuration represents the final state that the experimentalist is interested in measuring. Then, the amplitude that the vacuum evolves back into the vacuum in the presence of these sources is precisely the amplitude that the particles scatter from the chosen initial state into the chosen final state. This is illustrated below:



Motivated by these considerations, we define the vacuum–vacuum amplitude in the presence of the source field J:

$$\boxed{Z[J] \stackrel{\text{def}}{=} \lim_{T\to\infty} \langle 0|\hat{U}_J(+T,-T)|0\rangle.} \tag{1.3}$$

Here $|0\rangle$ is the vacuum state, and $\hat{U}_J$ is the time evolution operator in the presence of the source J. Note that

$$Z[0] = \lim_{T\to\infty} e^{-iE_0(2T)}, \tag{1.4}$$

where E_0 is the vacuum energy.This is another one of those singular normalization factors that will cancel when we compute physical quantities.

To compute this in terms of the path integral, we use the result of the previous chapter that the $i\epsilon$ prescription projects out the ground state. We therefore have

$$\boxed{Z[J] = \mathcal{N} \int d[\phi]\, e^{iS[\phi]+i\int J\phi},} \tag{1.5}$$

with the $i\epsilon$ prescription is understood, and $\mathcal{N}$ is a (singular) normalization factor. We need not specify the initial and final configurations in the path integral, since the ground state projection erases this information.[1] Eq. (1.5) is the starting point for the application of path integrals to quantum field theory.

[1]If we perform the path integral over field configurations with $\phi(\vec{x}, t_i) = \phi_i(\vec{x})$, $\phi(\vec{x}, t_f) = \phi_f(\vec{x})$,

If the source J is sufficiently weak, if makes sense to evaluate $Z[J]$ in an expansion in powers of J. Expanding the action in powers of J

$$\exp\left\{i\int d^4x\, J(x)\phi(x)\right\} = 1 + i\int d^4x\, J(x)\phi(x) + \frac{i^2}{2}\int d^4x d^4y\, J(x)\phi(x)J(y)\phi(y) + \cdots \tag{1.6}$$

and substituting into the path integral Eq. (1.5), we obtain

$$Z[J] = Z[0]\left(1 + i\int d^4x\, J(x)\langle\phi(x)\rangle + \frac{i^2}{2}\int d^4x d^4y\, J(x)J(y)\langle\phi(x)\phi(y)\rangle + \cdots\right), \tag{1.7}$$

where

$$\boxed{\langle\phi(x_1)\cdots\phi(x_n)\rangle \stackrel{\text{def}}{=} \frac{\displaystyle\int d[\phi]\, e^{iS[\phi]}\,\phi(x_1)\cdots\phi(x_n)}{\displaystyle\int d[\phi]\, e^{iS[\phi]}}.} \tag{1.8}$$

We can also summarize this by writing

$$\boxed{Z[J] = Z[0]\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\cdots\int d^4x_n\, J(x_1)\cdots J(x_n)\langle\phi(x_1)\cdots\phi(x_n)\rangle.} \tag{1.9}$$

This defines the **correlation functions**. Thus, one way of looking at $Z[J]$ is that it defines the correlation functions as the coefficients in the expansion in powers of J; we say that $Z[J]$ is the **generator** of the correlation functions. Note that the correlation functions are independent of the overall normalization of the path integral measure.

We now interpret the correlation functions defined above. We claim that they are precisely the time-ordered Green's functions familiar from the operator formalism:

$$\boxed{\langle\phi(x_1)\cdots\phi(x_n)\rangle = \langle 0|T\hat{\phi}_{\rm H}(x_1)\cdots\hat{\phi}_{\rm H}(x_n)|0\rangle,} \tag{1.10}$$

where $\hat{\phi}_{\rm H}(x)$ is the Heisenberg field operator. Another frequently-used notation for the Green's functions is

$$G^{(n)}(x_1,\ldots,x_n) = \langle 0|T\hat{\phi}_{\rm H}(x_1)\cdots\hat{\phi}_{\rm H}(x_n)|0\rangle. \tag{1.11}$$

then $\mathcal{N} \propto \Psi_0[\phi_f]\Psi_0^*[\phi_i]$, where Ψ_0 is the ground state wave functional. Therefore, the normalization factor does depend on the choice of boundary conditions on the path integral, but (as stated repeatedly) this drops out of physical quantities.

To prove Eq. (1.10), we go back to the case of quantum mechanics for notational simplicity. We first recall the definition of the Heisenberg picture. Heisenberg position operator $\hat{q}_H(t)$ is related to the Schrödinger picture operator $\hat{q}$ by

$$\hat{q}_{\rm H}(t) \stackrel{\rm def}{=} e^{+i\hat{H}t}\hat{q}e^{-i\hat{H}t}. \tag{1.12}$$

Also, the the Heisenberg position eigenstate

$$|q,t\rangle \stackrel{\rm def}{=} e^{+i\hat{H}t}|q\rangle \tag{1.13}$$

is time dependent, and satisfies

$$\hat{q}_{\rm H}(t)|q,t\rangle = q|q,t\rangle. \tag{1.14}$$

Using this notation, we can write the basic path integral formula as

$$\langle q_f, t_f|q_i, t_i\rangle = \langle q_f|e^{-i\hat{H}t_f}e^{+i\hat{H}t_i}|q_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} d[q]\, e^{iS[q]}. \tag{1.15}$$

To prove the connection between path integral correlation functions and time-ordered products, we first evaluate the time ordered product $\langle q_f, t_f|T\hat{q}_{\rm H}(t_1)\hat{q}_{\rm H}(t_2)|q_i, t_i\rangle$ in terms of a path integral. We assume without loss of generality that $t_2 > t_1$. Then

$$\langle q_f, t_f|T\hat{q}_{\rm H}(t_1)\hat{q}_{\rm H}(t_2)|q_i, t_i\rangle = \langle q_f, t_f|\hat{q}_{\rm H}(t_2)\hat{q}_{\rm H}(t_1)|q_i, t_i\rangle. \tag{1.16}$$

Now insert a complete set of Heisenberg states after each operator:

$$\int dq\, |q,t\rangle\langle q,t| = e^{+i\hat{H}t}\left(\int dq\, |q\rangle\langle q|\right)e^{-i\hat{H}t} = 1. \tag{1.17}$$

This gives

$$\begin{aligned}\langle q_f, t_f|T\hat{q}_{\rm H}(t_1)\hat{q}_{\rm H}(t_2)|q_i, t_i\rangle &= \int dq_2 \int dq_1\, \langle q_f, t_f|\hat{q}_{\rm H}(t_2)|q_2, t_2\rangle \\ &\qquad \times \langle q_2, t_2|q_1, t_1\rangle\langle q_1, t_1|\hat{q}_{\rm H}(t_1)|q_i, t_i\rangle \\ &= \int dq_2\, q_2 \int dq_1\, q_1\, \langle q_f, t_f|q_2, t_2\rangle \\ &\qquad \times \langle q_2, t_2|q_1, t_1\rangle\langle q_1, t_1|q_i, t_i\rangle. \end{aligned} \tag{1.18}$$

The three matrix elements in the above expression are each given by a path integral:

$$\begin{aligned}\langle q_f, t_f|T\hat{q}_{\rm H}(t_1)\hat{q}_{\rm H}(t_2)|q_i, t_i\rangle &= \int dq_2\, q_2 \int dq_1\, q_1 \\ &\qquad \times C\int_{q(t_2)=q_2}^{q(t_f)=q_f} d[q]\, e^{iS[q]}\, C\int_{q(t_1)=q_1}^{q(t_2)=q_2} d[q]\, e^{iS[q]}\, C\int_{q(t_i)=q_i}^{q(t_1)=q_1} d[q]\, e^{iS[q]} \\ &= C\int_{q(t_i)=q_i}^{q(t_f)=q_f} d[q]\, e^{iS[q]}\, q(t_1)q(t_2), \end{aligned} \tag{1.19}$$

where the last line uses the path integral composition formula discussed in the previous chapter. Note that we considered the case $t_2 > t_1$; if we had considered $t_2 < t_1$ instead, the same steps would also lead to the right-hand side of Eq. (1.19). This is as it should be, since $T\hat{q}_{\rm H}(t_1)\hat{q}_{\rm H}(t_2) = T\hat{q}_{\rm H}(t_2)\hat{q}_{\rm H}(t_1)$. Now, the $i\epsilon$ prescription projects out the ground state in the usual way, and we obtain

$$\langle 0|T\hat{q}_{\rm H}(t_1)\hat{q}_{\rm H}(t_2)|0\rangle = C\int d[q]\, e^{iS[q]}\, q(t_1)q(t_2). \tag{1.20}$$

To eliminate the divergent normalization factor, we divide out by the path integral with no operator insertions:

$$\langle 0|T\hat{q}_{\rm H}(t_1)\hat{q}_{\rm H}(t_2)|0\rangle = \frac{\displaystyle\int d[q]\, e^{iS[q]}\, q(t_1)q(t_2)}{\displaystyle\int d[q]\, e^{iS[q]}}. \tag{1.21}$$

This argument obviously generalizes to products of more than two operators, and the field theory generalization immediately gives Eq. (1.10).

We can also write Eq. (1.10) using functional derivatives. We can write

$$\frac{1}{Z[0]}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0} = i^n\langle\phi(x_1)\cdots\phi(x_n)\rangle. \tag{1.22}$$

That is, the correlation functions are defined by functional differentiation of the generating functional with respect to the sources. We can also define correlation functions for operators other than ϕ (*e.g.* ϕ^2 or $\partial_\mu\phi$) by adding additional sources for them in the Lagrangian. For operators that are already present in the Lagrangian, this is equivalent to allowing the coupling constants to depend on spacetime. For example, the operator $\frac{1}{2}\phi^2$ can be defined by promoting the mass term to a spacetime dependent source $m^2 \to \mu^2(x)$. The generating functional can then be written as $Z[J,\mu^2]$. We can then define *e.g.*

$$\frac{1}{Z[0,m^2]}\frac{\delta}{\delta\mu^2(x)}Z[J,\mu^2]\bigg|_{J=0,\,\mu^2=m^2} = -\frac{i}{2}\langle\phi^2(x)\rangle. \tag{1.23}$$

Promoting coupling constants to sources in this way is a very useful way to keep track of the consequences of symmetries, as we will discuss later.

2 Free Field Theory

Up to now, we have been concerned with establishing the connection between the path integral and the operator formulation of quantum mechanics. We now compute the path integral for a free field theory. Although free field theory is physically trivial, it is important as the starting point for weak coupling perturbation theory. This section also marks the point where we begin to break free of the operator formulation and use the path integral on its own.

The action is

$$S_0 = \int d^4x \left[\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2\right]. \tag{2.1}$$

The subscript 0 reminds us that this is a free theory. We will compute

$$Z_0[J] = \int d[\phi]\, e^{iS[\phi]+i\int J\phi}, \tag{2.2}$$

which gives us the complete correlation functions of the theory. (Without any source terms, the path integral is just a divergent number $Z_0[0]$!)

We replace the spacetime continuum by a hypercubic lattice to make everything well-defined:

$$x = a \cdot (n_0, n_1, n_2, n_3), \tag{2.3}$$

where n_0, n_1, n_2, n_3 are integers and a is the lattice spacing. This treats time and space more symmetrically than the spatial lattice used in the last chapter. Discretizing the theory is a drastic modification of the theory at the scale a, but it is not expected to change the physics at distance scales much larger than a. In the language of the next chapter, discretizing the theory is one way to provide the theory with a **short distance cutoff**.

To define the discretized action, it is convenient to rewrite the continuum action as

$$S_0[\phi] = \int d^4x \, \frac{1}{2}\left[\phi(-\Box)\phi - m^2\phi^2\right]. \tag{2.4}$$

Note that $\Box = \partial^\mu\partial_\mu$ is a linear operator, and can therefore be thought of as a kind of matrix. In fact, in the discretized theory the derivative operator ∂_μ is replaced by the difference operator Δ_μ, which really is a matrix:

$$\begin{aligned}(\Delta_\mu\phi)_x &= \frac{\phi_{x+\mu} - \phi_{x-\mu}}{2a} \\ &= \sum_y (\Delta_\mu)_{xy}\phi_y,\end{aligned} \tag{2.5}$$

where

$$(\Delta_\mu)_{xy} = \frac{1}{2a}\left(\delta_{x+\mu,y} - \delta_{x-\mu,y}\right). \tag{2.6}$$

Here μ runs over unit lattice 4-vectors $(a,0,0,0),\ldots,(0,0,0,a)$. (Note we have used a symmetric definition of the derivative, so that Δ_μ is a symmetric matrix.) We can therefore write the discretized action as

$$S_0[\phi] = \sum_x a^4 \left[-\frac{1}{2} \sum_y \phi_x (\Delta^2)_{xy} \phi_y - \frac{m^2}{2} \phi_x^2 \right]$$
$$= \sum_{x,y} \frac{a^4}{2} \phi_x A_{xy} \phi_y, \tag{2.7}$$

where

$$A_{xy} = -(\Delta^2)_{xy} - m^2 \delta_{xy}. \tag{2.8}$$

is a symmetric matrix.

Our job is therefore to evaluate

$$Z_0[J] = \int \left(\prod_x d\phi_x \right) \exp\left\{ \frac{i}{2} \sum_{x,y} a^4 \phi_x A_{xy} \phi_y + i \sum_x a^4 J_x \phi_x \right\}. \tag{2.9}$$

This is just a generalized Gaussian integral with a matrix defining the quadratic term. To evaluate it, note that A is a symmetric matrix, so it can be diagonalized by an orthogonal transformation.[2] That is, there exists a matrix R with $R^T = R^{-1}$ such that

$$\tilde{A} = RAR^T = \text{diagonal}. \tag{2.10}$$

If we define

$$\tilde{\phi} = R\phi, \qquad \tilde{J} = RJ, \tag{2.11}$$

we have

$$\sum_{x,y} \tfrac{1}{2} \phi_x A_{xy} \phi_y + \sum_x J_x \phi_x = \sum_k \left(\tfrac{1}{2} \lambda_k \tilde{\phi}_k^2 + \tilde{J}_k \tilde{\phi}_k \right), \tag{2.12}$$

where λ_k are the eigenvalues of A. We now change integration variables from ϕ to $\tilde{\phi}$. Because R is an orthogonal transformation, the measure is invariant

$$\prod_x d\phi_x = \prod_k d\tilde{\phi}_k. \tag{2.13}$$

[2]Note that if we had used a non-symmetric discretization of the derivative, then A will not be a symmetric matrix, but it is only the symmetric part of A that contributes to the action.

We can set $a = 1$, which corresponds to measuring all distances in lattice units. We can always restore the dependence on a by dimensional analysis. We then have

$$Z_0[J] = \int \left(\prod_k d\tilde{\phi}_k \right) \exp\left\{ \sum_k \left(\frac{i\lambda_k}{2} \tilde{\phi}_k^2 + i\tilde{J}_k \tilde{\phi}_k \right) \right\}. \tag{2.14}$$

In terms of these variables, the functional integral is just a product of Gaussian integrals:

$$Z_0[J] = \prod_k \left(\int d\phi_k \, \exp\left\{ \frac{i\lambda_k}{2} \tilde{\phi}_k^2 + i\tilde{J}_k \tilde{\phi}_k \right\} \right). \tag{2.15}$$

These are well-defined Gaussian integrals provided that $\mathrm{Im}(\lambda_k) > 0$ for *all* λ_k. We will verify below that the $i\epsilon$ prescription gives the eigenvalues a positive imaginary part, as required. We then obtain

$$Z_0[J] = \prod_k \left[\left(\frac{2\pi i}{\lambda_k} \right)^{1/2} \exp\left\{ -\frac{i}{2\lambda_k} \tilde{J}_k^2 \right\} \right]. \tag{2.16}$$

We can write this in a general basis by noting that

$$\prod_k \left(\frac{1}{\lambda_k} \right)^{1/2} = [\mathrm{Det}(A)]^{-1/2}, \tag{2.17}$$

$$\prod_k \exp\left\{ -\frac{i}{2\lambda_k} \tilde{J}_k^2 \right\} = \exp\left\{ -\frac{i}{2} \sum_{x,y} J_x (A^{-1})_{xy} J_y \right\}. \tag{2.18}$$

We then have

$$\boxed{Z_0[J] = \mathcal{N} \left[\mathrm{Det}(-\Delta^2 - m^2) \right]^{-1/2} \exp\left\{ -\frac{i}{2} \sum_{x,y} J_x (-\Delta^2 - m^2)^{-1}_{xy} J_y \right\}} \tag{2.19}$$

where

$$\mathcal{N} = \prod_x \left(\frac{2\pi i}{a^4} \right)^{1/2} \tag{2.20}$$

is a (divergent) normalization factor. In fact, both $\mathcal{N}$ and $\mathrm{Det}(A)$ are constants (independent of J), and therefore do not affect the correlation functions. (Note that we have restored factors of a by dimensional analysis.)

We now verify that the $i\epsilon$ prescription gives a positive imaginary part to all of the eigenvalues of A. We will do this using a continuum notation for simplicity, but it should be clear that the results hold in the discretized case as well. The $i\epsilon$ prescription tells us to make the replacement

$$t \to (1 - i\epsilon)t, \tag{2.21}$$

for $\epsilon > 0$. To linear order in ϵ this gives

$$\int d^4x \to (1 - i\epsilon) \int d^4x, \tag{2.22}$$

$$\dot{\phi}^2 \to (1 + 2i\epsilon)\dot{\phi}^2. \tag{2.23}$$

Therefore, the action becomes

$$\begin{aligned} S_0[\phi] &= \int d^4x \, \frac{1}{2} \left[\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2 \right] \\ &\to S_0[\phi] + i\epsilon \int d^4x \, \frac{1}{2} \left[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \right]. \end{aligned} \tag{2.24}$$

The integrand in the last term is positive definite, which shows that all the eigenvalues of A have positive imaginary parts, as required.

In continuum notation, we can write this as

$$\boxed{Z_0[J] = \mathcal{N} \left[\mathrm{Det}(-\Box - m^2) \right]^{-1/2} \exp\left\{ -\frac{i}{2} \sum_{x,y} J_x (-\Box - m^2)^{-1}_{xy} J_y \right\}} \tag{2.25}$$

To understand the meaning of the inverse of the operator $-\Box - m^2$ it is useful to go to momentum space. We can define momentum-space by

$$\tilde{\phi}(k) \stackrel{\text{def}}{=} \int d^4x \, e^{ik\cdot x} \phi(x). \tag{2.26}$$

Note that $\phi(x)$ is real, which implies that

$$\tilde{\phi}^\dagger(k) = \tilde{\phi}(-k). \tag{2.27}$$

In terms of momentum-space fields, the free action with the $i\epsilon$ prescription becomes

$$\begin{aligned} S_0[\phi] \to &\int \frac{d^4k}{(2\pi)^4} \, \frac{1}{2} \tilde{\phi}^\dagger(k) \left(k^2 - m^2 \right) \tilde{\phi}(k) \\ &+ i\epsilon \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{2} \tilde{\phi}^\dagger(k) \left(k_0^2 + \vec{k}^2 + m^2 \right) \tilde{\phi}(k) \\ = &\int \frac{d^4k}{(2\pi)^4} \, \frac{1}{2} \tilde{\phi}^\dagger(k) \left(k^2 - m^2 + i\epsilon \right) \tilde{\phi}(k). \end{aligned} \tag{2.28}$$

In the last line, we have replaced $\epsilon(k_0^2 + \vec{k}^2 + m^2)$ by ϵ. This is legitimate because ϵ just stands for an infinitesimal positive quantity that is taken to zero at the end of the calculation.

Eq. (2.28) shows that the momentum-space fields diagonalize the kinetic term. Note also that the momentum-space expression shows clearly that the imaginary part is nonvanishing for all eigenvalues.

The momentum space expressions above also tell us how to interpret the inverse of the kinetic operator in the continuum limit. The continuous version of Eq. (2.19) is

$$Z_0[J] = Z_0[0] \exp\left\{ -\frac{i}{2} \int d^4x d^4y \, J(x) \Delta(x, y) J(y) \right\}, \tag{2.29}$$

where $\Delta(x_1, x_2)$ is the continuous version of the inverse matrix for the kinetic term:

$$\left(-\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} - m^2 + i\epsilon \right) \Delta(x, y) = \delta^4(x - y). \tag{2.30}$$

That is, $\Delta(x, y)$ is a Green's function for the free equations of motion. In general, a Green's function requires the specification of boundary conditions to be well-defined. In the present approach, the $i\epsilon$ prescription tells us that the appropriate Green's function is

$$\boxed{\Delta(x, y) = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x-y)} \frac{1}{k^2 - m^2 + i\epsilon}.} \tag{2.31}$$

This is exactly the Feynman propagator encountered in the operator formalism. Our final expression for $Z_0[J]$ in the continuum is therefore

$$\boxed{Z_0[J] = Z_0[0] \exp\left\{ -\frac{i}{2} \int d^4x d^4y \, J(x) \Delta(x, y) J(y) \right\}.} \tag{2.32}$$

This equation expresses all the correlation functions for free field theory in a very compact way.

2.1 Diagrammatic Expansion

Let us obtain an explicit formula for the correlation functions of free field theory. All we have to do is expand $Z_0[J]$ in powers of J and use Eq. (1.9) to read off the correlation functions. Since only even powers of J appear, we have

$$\begin{aligned} Z_0[J] = Z_0[0] \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \right)^n \int & d^4x_1 \cdots d^4x_{2n} \, J(x_1) \cdots J(x_{2n}) \\ & \times \Delta(x_1, x_2) \cdots \Delta(x_{2n-1}, x_{2n}). \end{aligned} \tag{2.33}$$

To read off the correlation functions, we must remember that they are completely symmetric functions of their arguments. Comparing with Eq. (1.9) gives

$$\frac{i^{2n}}{(2n)!}\langle\phi(x_1)\cdots\phi(x_{2n})\rangle_0 = \frac{1}{n!}\left(-\frac{i}{2}\right)^n [\Delta(x_1,x_2)\cdots\Delta(x_{2n-1},x_{2n})]\,. \tag{2.34}$$

Here the square brackets on the right-hand side tell us to symmetrize in the arguments $x_1, \ldots x_{2n}$. We therefore have

$$\langle\phi(x_1)\cdots\phi(x_{2n})\rangle_0 = \frac{1}{n!}\left(\frac{i}{2}\right)^n \sum_\sigma \Delta(x_{\sigma_1},x_{\sigma_2})\cdots\Delta(x_{\sigma_{2n-1}},x_{\sigma_{2n}}), \tag{2.35}$$

where the sum over σ runs over the $(2n)!$ permutations of $1,\ldots,2n$.

Eq. (2.35) is not the most convenient form of the answer, because many terms in the sum are the same. The order of the Δ's does not matter and $\Delta(x,y) = \Delta(y,x)$. The distinct terms correspond precisely to the possible pairings of the indices $1,\ldots,2n$. For each distinct term, there are 2^n permutations corresponding to interchanging the order of the indices on the Δ's in all possible ways, and $n!$ ways of reordering the Δ's. Therefore, we must multiply each distinct term by $2^n n!$. This gives

$$\langle\phi(x_1)\cdots\phi(x_{2n})\rangle_0 = {\sum_\sigma}' i\Delta(x_{\sigma_1},x_{\sigma_2})\cdots i\Delta(x_{\sigma_{2n-1}},x_{\sigma_{2n}}), \tag{2.36}$$

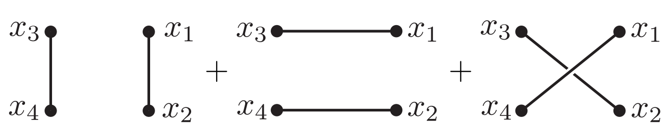
where the sum is now over the possible pairings of $1,2,\ldots,2n$. This is **Wick's theorem** derived in terms of path integrals.

We can write this in diagrammatic language by writing a dot for each position $x_1,\ldots,x_{2n}$ and denoting a Feynman propagator $i\Delta(x,y)$ by a line connecting the dots x and y:

$$= i\Delta(x,y). \tag{2.37}$$

The possible pairings just correspond to the possible "contractions," *i.e.* the distinct ways of connecting the dots. For example,

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_0 =$$



$$= i\Delta(x_1,x_2)i\Delta(x_3,x_4) + i\Delta(x_1,x_3)i\Delta(x_2,x_4) + i\Delta(x_1,x_4)i\Delta(x_2,x_3). \tag{2.38}$$

We see that the Feynman rules for free field theory emerge very elegantly from the path integral.

3 Weak Coupling Perturbation Theory

We now show how to evaluate the correlation functions for an interacting theory by expanding about the free limit. We write the action as

$$S[\phi] = S_0[\phi] + S_{\rm int}[\phi], \tag{3.1}$$

where S_0 is the free action and $S_{\rm int}$ contains the interaction terms, which we treat as a perturbation. For definiteness, we take

$$S_{\rm int}[\phi] = -\frac{\lambda}{4!}\int d^4x\, \phi^4. \tag{3.2}$$

We expect that perturbing in $S_{\rm int}$ will be justified as long as λ is sufficiently small.

To evaluate the correlation functions in perturbation theory, we start with the definition Eq. (1.8) for the correlation functions

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\displaystyle\int d[\phi]\, e^{iS[\phi]}\phi(x_1)\cdots\phi(x_n)}{\displaystyle\int d[\phi]\, e^{iS[\phi]}} \tag{3.3}$$

and expand both the numerator and denominator in powers of $S_{\rm int}$. Up to $\mathcal{O}(\lambda^2)$ corrections, this gives

$$\begin{aligned}\text{denominator} &= \int d[\phi]\, e^{iS_0[\phi]}\left\{1 + iS_{\rm int}[\phi] + \mathcal{O}(\lambda^2)\right\}\\ &= Z_0[0]\left\{1 + i\left(-\frac{\lambda}{4!}\right)\int d^4y\, \langle \phi^4(y)\rangle_0 + \mathcal{O}(\lambda^2)\right\}.\end{aligned} \tag{3.4}$$

Note that the terms containing powers of $S_{\rm int}$ are just correlation functions in the *free* theory. The same thing happens for the numerator:

$$\begin{aligned}\text{numerator} &= \int d[\phi]\, e^{iS_0[\phi]}\phi(x_1)\cdots\phi(x_n)\left\{1 + iS_{\rm int}[\phi] + \mathcal{O}(\lambda^2)\right\}\\ &= Z_0[0]\bigg\{\langle\phi(x_1)\cdots\phi(x_n)\rangle_0\\ &\qquad + i\left(-\frac{\lambda}{4!}\right)\int d^4y\, \langle\phi(x_1)\cdots\phi(x_n)\phi^4(y)\rangle_0 + \mathcal{O}(\lambda^2)\bigg\}.\end{aligned} \tag{3.5}$$

Let us evaluate these expressions explicitly for the 2-point function. Dividing Eq. (3.5) by Eq. (3.4), we obtain

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\rangle &= \langle\phi(x_1)\phi(x_2)\rangle_0 + i\left(-\frac{\lambda}{4!}\right)\int d^4y\, \langle\phi(x_1)\phi(x_2)\phi^4(y)\rangle_0\\ &\qquad - i\langle\phi(x_1)\phi(x_2)\rangle_0\left(-\frac{\lambda}{4!}\right)\int d^4y\, \langle\phi^4(y)\rangle_0 + \mathcal{O}(\lambda^2).\end{aligned} \tag{3.6}$$

The free correlation functions are easily evaluated using the results above. For example,

$$\langle \phi^4(y) \rangle_0 = 3\left[i\Delta(y,y)\right]^2. \tag{3.7}$$

The factor of 3 comes from the 3 possible contractions. Similarly,

$$\begin{aligned}\langle \phi(x_1)\phi(x_2)\phi^4(y) \rangle_0 = 3i\Delta(x_1,x_2)\left[i\Delta(y,y)\right]^2 \\ + 12 i\Delta(x_1,y) i\Delta(x_2,y) i\Delta(y,y).\end{aligned} \tag{3.8}$$

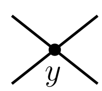
Again, the factors count the number of contractions. (We will discuss these factors below, so we do not dwell on it now.) When we substitute into Eq. (3.6), we see that the contribution from Eq. (3.7) cancels the first term in Eq. (3.8), and we are left with

$$\langle \phi(x_1)\phi(x_2) \rangle = \langle \phi(x_1)\phi(x_2) \rangle_0 - \frac{i\lambda}{2}\int d^4y\, i\Delta(x_1,y) i\Delta(x_2,y) i\Delta(y,y) + \mathcal{O}(\lambda^2). \tag{3.9}$$

We now derive the diagrammatic rules to generate this expansion. As above, we denote a contraction between fields $\phi(x)$ and $\phi(y)$ by by a line connecting the points x and y. Each such contraction gives a Feynman propagator, so we write

$$= i\Delta(x,y). \tag{3.10}$$

In addition, there are contractions involving fields ϕ coming from the ϕ^4 interaction terms. Each such term comes with a factor of $-i\lambda$, so we write each such term as a vertex



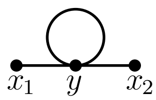
$$= -i\lambda. \tag{3.11}$$

We then integrate over the positions of all of the vertices. These rules do not take into account a combinatoric factor that appears as an overall coefficient for each diagram. In a diagram with V vertices, there is a factor of $1/4!$ from each vertex, and a factor of $1/V!$ from the fact that the contribution comes from the $(S_{\rm int})^V$ term in the expansion of $\exp\{iS_{\rm int}\}$. Finally, we must multiply by the number of contractions C that give rise to the same diagram. We therefore multiply each diagram by

$$S = \left(\frac{1}{4!}\right)^V \frac{C}{V!}. \tag{3.12}$$

S is sometimes called the 'symmetry factor' of the diagram. Some books give general rules for finding S, but it is usually less confusing just to count the contractions explicitly.

This notation allows us to easily find all contributions to the numerator and denominator of Eq. (1.8) at any given order in the coupling constant expansion. For example, let us consider again the evaluation of the 2-point function. The numerator is the sum of all diagrams with two external points. At $\mathcal{O}(\lambda)$, there are only two diagrams:

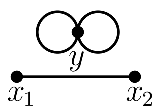


$$= \frac{-i\lambda}{2} \int d^4y \, i\Delta(x_1, y) i\Delta(x_2, y) i\Delta(y, y), \tag{3.13}$$

with

$$S = \frac{1}{4!} \cdot 4 \cdot 3 = \frac{1}{2}, \tag{3.14}$$

and

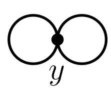


$$= \frac{-i\lambda}{6} i\Delta(x_1, x_2) \int d^4y \left[i\Delta(y, y) \right]^2, \tag{3.15}$$

with

$$S = \frac{1}{4!} \cdot 3 = \frac{1}{6}. \tag{3.16}$$

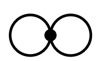
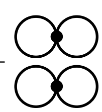
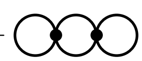
The denominator corresponds to diagrams with no external points. Such graphs are often called **vacuum graphs**. At $\mathcal{O}(\lambda)$, the only diagram is



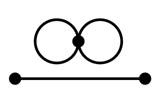
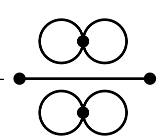
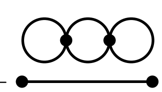
$$= \frac{-i\lambda}{6} \int d^4y \left[i\Delta(y, y) \right]^2, \tag{3.17}$$

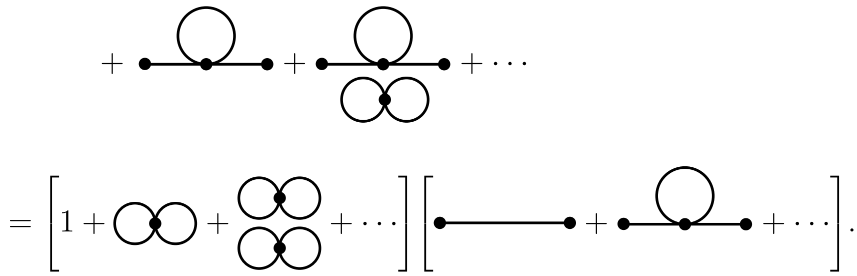
with the symmetry factor the same as in Eq. (3.15).

We can now see that the denominator exactly cancels all the diagrams such as Eq. (3.15) that have vacuum subdiagrams. To see this, note that the denominator is the sum of all vacuum graphs:

denominator = 1 +  +  +  + ··· (3.18)

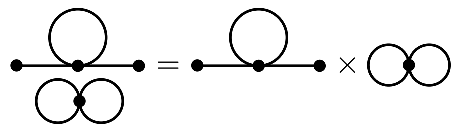
(Why is there no vacuum diagram ?) For the 2-point function the numerator can be written

numerator = +  +  +  + ···



(3.19)

The reason is that *e.g.*



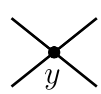
(3.20)

The nontrivial part of this statement is that the symmetry factor of the diagram on the left-hand side is the product of the symmetry factors of the diagrams on the right-hand side. We will prove below that the cancelation of disconnected diagrams holds to all orders in the perturbative expansion. Assuming this result for a moment, we can summarize the **position space Feynman rules** for computing $\langle\phi(x_1)\cdots\phi(x_n)\rangle$:

- Draw all diagrams with n external points $x_1, \ldots, x_n$ with no vacuum subgraphs.
- Associate a factor

$$= i\Delta(x, y) \tag{3.21}$$

for each propagator, and a factor



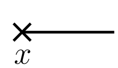
$$= -i\lambda \tag{3.22}$$

for each vertex.

- Integrate over the positions of all vertices.
- Multiply by the symmetry factor given by Eq. (3.12).

3.1 Connected Diagrams

To understand the general relation between connected and disconnected diagrams, it is useful to view the generating functional $Z[J]$ as the sum of all diagrams in the presence of a nonzero source $J(x)$. This source gives rise to a position-space vertex



$$= iJ(x). \tag{3.23}$$

With this identification, $Z[J]$ is simply the sum of all graphs with no external legs in the presence of the source; the source J creates what used to be the external legs. We summarize this simply by

$$Z[J] = \sum \text{diagrams}, \tag{3.24}$$

where 'diagrams' refers to all diagrams without vacuum subgraphs, but including disconnected pieces.

With this notation, we can state our result:

$$\boxed{Z[J] = \exp\left\{\sum \text{connected diagrams}\right\}.} \tag{3.25}$$

Note that since disconnected diagrams are products of connected diagrams, every disconnected diagram appears in the expansion of the exponential on the right-hand side. The nontrivial part of Eq. (3.25) is that the disconnected diagrams are generated with the correct coefficients. Note also that the generating functional for free field theory has this form.

There is a very pretty proof of this that is based on the **replica trick**. Consider a new Lagrangian that consists of N identical copies of the theory we are interested in, where the different theories do not interact between each other. That is, we consider the generating functional

$$Z_N[J] = \int d[\phi_1] \cdots d[\phi_N]\, e^{iS[\phi_1]+i\int J\phi_1} \cdots e^{iS[\phi_n]+i\int J\phi_N}. \tag{3.26}$$

This is related to the generating functional of our original theory by

$$Z_N[J] = (Z[J])^N. \tag{3.27}$$

The Feynman rules for the new theory are also closely related to the original theory. The only difference is that every field line or vertex can be any one of the N fields $\phi_1, \ldots, \phi_N$. Therefore, every connected diagram is of order N, since once we choose the identity of one of the lines, all the other lines and vertices must involve the same field. A diagram with n disconnected pieces is proportional to N^n, since we can choose among N fields for each disconnected piece. We see that the connected diagrams are those proportional to N. From Eq. (3.27) we can read off the term proportional to N:

$$Z_N[J] = e^{N \ln Z[J]} = 1 + N \ln Z[J] + \mathcal{O}(N^2). \tag{3.28}$$

We see that the connected diagrams sum to $\ln Z[J]$, which is equivalent to Eq. (3.25).

This result immediately implies the cancellation of disconnected vacuum diagrams in the expansion of $Z[J]$. The reason is simply that

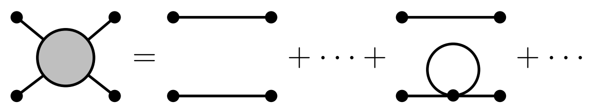
$$Z[0] = \exp\{\text{connected vacuum graphs}\}\,, \tag{3.29}$$

while

$$\begin{aligned} Z[J] &= \exp\{\text{connected vacuum graphs} + \text{connected non-vacuum graphs}\} \\ &= Z[0]\exp\{\text{connected non-vacuum graphs}\}\,, \end{aligned} \tag{3.30}$$

where ‘connected non-vacuum graphs’ refers to graphs with at least one source vertex. We see that dividing by $Z[0]$ precisely cancels the vacuum graphs.

Although the diagrams that contribute to the correlation functions do not include vacuum subdiagrams, it does include other kinds of disconnected graphs. For example,



(3.31)

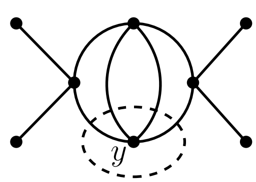
The exponentiation of the disconnected diagrams means that these diagrams are trivially determined from the connected graphs. In any case, we will see that the connected graphs contain all the information that we need to do physics.

3.2 Feynman Rules in Momentum Space

For practical calculations, it is simpler to formulate the Feynman rules in momentum space. To translate the rules given above into momentum space, we just write the position space propagator in terms of momentum space:

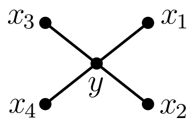
$$\Delta(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}, \tag{3.32}$$

Consider a vertex at position y that appears somewhere inside of an arbitrary diagram, as in the example below:



(3.33)

The part of the diagram that involves y can be written as

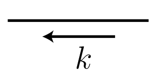


$$
\begin{aligned}
&= -i\lambda \int d^4y\, i\Delta(x_1, y) i\Delta(x_2, y) i\Delta(x_3, y) i\Delta(x_4, y) \\
&= -i\lambda \int d^4y \int \frac{d^4k_1}{(2\pi)^4} \frac{ie^{-ik_1\cdot(x_1-y)}}{k_1^2 - m^2} \cdots \int \frac{d^4k_4}{(2\pi)^4} \frac{ie^{-ik_4\cdot(x_4-y)}}{k_4^2 - m^2} \\
&= -i\lambda \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_4}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \cdots + k_4) \\
&\qquad \times \frac{i}{k_1^2 - m^2} \cdots \frac{i}{k_4^2 - m^2} \\
&\qquad \times e^{-ik_1\cdot x_1} \cdots e^{-ik_4\cdot x_4},
\end{aligned} \tag{3.34}
$$

where we have performed the y integral. We interpret $k_1, \ldots, k_4$ as momenta flowing into the vertex. In this expression, $x_1, \ldots, x_4$ may be either external or internal points. Also, some of $x_1, \ldots, x_4$ may be identical because they are internal points at the same vertex (as in the diagram shown in Eq. (3.33)). If x_1 (for example) is an external point, then the factor $e^{-ik_1\cdot x_1}$ remains as an external 'wavefunction' factor. If x_1 is an internal point, then the factor of $e^{-ik_1\cdot x_1}$ will be absorbed in the integral over x_1, similar to the expression Eq. (3.34) for the integral over y. Notice that the exponent has the 'wrong' sign compared to the example above. This is accounted for by noting that the momentum k_1 flows away from the point x_1, so the momentum flowing *into* the point x_1 is negative.

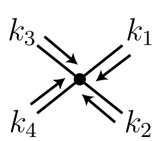
From these considerations, we obtain a new set of Feynman rules for computing $\langle\phi(x_1)\cdots\phi(x_n)\rangle$:

- Draw all diagrams with n external points $x_1, \ldots, x_n$ with no vacuum subgraphs.
- Associate a factor



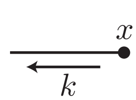
$$
= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \tag{3.35}
$$

for each propagator, a factor



$$
= -i\lambda(2\pi)^4\delta^4(k_1 + k_2 + k_3 + k_4) \tag{3.36}
$$

for each vertex, and a factor



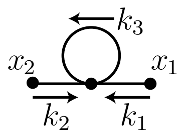
$$
= e^{-ik\cdot x} \tag{3.37}
$$

for each external point.

- Integrate over all internal momenta.
- Multiply by the symmetry factor.

These Feynman rules can be simplified further, since some of the momentum integrals are trivial because of the delta function. To see this, let us do some examples.

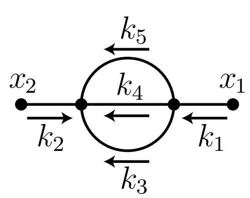
First, consider the contribution to the 2-point correlation function



$$= \frac{-i\lambda}{2} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + k_2 + k_3 - k_3)$$
$$\times\, e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2}$$
$$\times \frac{i}{k_1^2 - m^2} \frac{i}{k_2^2 - m^2} \frac{i}{k_3^2 - m^2}$$
$$= \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + k_2)$$
$$\times\, e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} \frac{i}{k_1^2 - m^2} \frac{i}{k_2^2 - m^2}$$
$$\times \left[\frac{-i\lambda}{2} \int \frac{d^4k_3}{(2\pi)^4} \frac{i}{k_3^2 - m^2} \right]. \tag{3.38}$$

Notice that the factors in the first two lines of Eq. (3.38) will be present in any contribution to the 2-point function: there will always be an integral over the external momenta k_1 and k_2, there will always be a factor of $e^{-ik_1\cdot x_2} e^{-ik_2\cdot x_2}$ for the external points, and there will always be a delta function that enforces overall energy and momentum conservation. It is only the last term in brackets in Eq. (3.38) that is special to this contribution to the 2-point function.

Let us amplify these points by doing another example.



$$= \frac{(-i\lambda)^2}{6} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_5}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 - k_3 - k_4 - k_5)$$
$$\times (2\pi)^4 \delta^4(k_2 + k_3 + k_4 + k_5) e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2}$$
$$\times \frac{i}{k_1^2 - m^2} \cdots \frac{i}{k_5^2 - m^2}$$
$$= \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + k_2)$$

$$\times e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} \frac{i}{k_1^2 - m^2} \frac{i}{k_2^2 - m^2}$$
$$\times \left[\frac{(-i\lambda)^2}{6} \int \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4} \frac{i}{k_3^2 - m^2} \frac{i}{k_4^2 - m^2} \right.$$
$$\left. \times \frac{i}{(k_1 - k_2 - k_3)^2 - m^2} \right]. \tag{3.39}$$

Note that when the k_5 integral was performed using one of the delta functions, the left over delta function just enforces overall energy and momentum conservation. Again, we see explicitly the factors that are present in any contribution to the 2-point function.

It is convenient to factor out the factors common to every diagram by defining

$$\begin{aligned} \langle \phi(x_1) \cdots \phi(x_n) \rangle &= G^{(n)}(x_1, \ldots, x_n) \\ &=: \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} e^{-ik_1 \cdots x_1} \cdots e^{-ik_n\cdot x_n} \\ &\qquad \times (2\pi)^4 \delta^4(k_1 + \cdots + k_n) \tilde{G}^{(n)}(k_1, \ldots, k_n). \end{aligned} \tag{3.40}$$

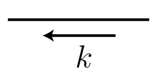
In other words, $\tilde{G}^{(n)}$ times a delta function is the Fourier transform of $G^{(n)}$. $\tilde{G}^{(n)}$ is called the **momentum-space Green's function**. The inverse Fourier transform gives

$$\boxed{\begin{aligned} &(2\pi)^4 \delta^4(k_1 + \cdots + k_n) \tilde{G}^{(n)}(k_1, \ldots, k_n) \\ &\qquad = \int d^4x_1 \cdots d^4x_n \, e^{ik_1\cdot x_1} \cdots e^{ik_n\cdot x_n} G^{(n)}(x_1, \ldots, x_n). \end{aligned}} \tag{3.41}$$

We can now state the **momentum-space Feynman rules** for the momentum-space Green's function $\tilde{G}^{(n)}$. (These are the rules in the form that Feynman originally wrote them.)

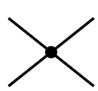
• Draw all diagrams with n external momenta $k_1, \ldots, k_n$ with no vacuum subgraphs. Assign momenta to all internal lines, enforcing momentum conservation at each vertex.

• Associate a factor



$$= \frac{i}{k^2 - m^2 + i\epsilon} \tag{3.42}$$

for each propagator, and a factor

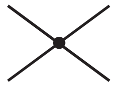
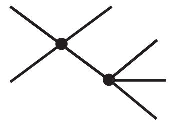


$$= -i\lambda \tag{3.43}$$

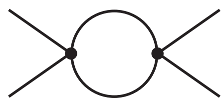
for each vertex.

- Integrate over all independent internal momenta.
- Multiply by the symmetry factor.

We close this section by defining some standard terminology. For graphs such as

 and 

the momentum in every internal lines is completely fixed in terms of the external momentum by momentum conservation at each vertex. Such diagrams are called **tree diagrams**, since they have the topological structure of trees. For graphs such as

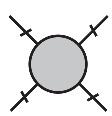
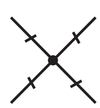


the momentum in the internal lines is not completely fixed by the external momenta, and some momenta must be integrated over. These graphs are called **loop diagrams**, since their topological structure involves at least one loop.

Also, note that the Green's function $\tilde{G}^{(n)}$ defined above includes a propagator for each internal line. It is sometimes useful to work with the amputated Green's function, where these factors are removed:

$$\tilde{G}^{(n)}(k_1, \ldots, k_n) = \frac{i}{k_1^2 - m^2} \cdots \frac{i}{k_n^2 - m^2} \tilde{G}_{\text{amp}}(k_1, \ldots, k_n). \tag{3.44}$$

This is a fairly trivial difference, and in fact the distinction between these two types of diagrams is often blurred in the research literature. In these lectures, we will always denote an amputated Green's function by putting a small slash through the external lines, *e.g.*

$\tilde{G}^{(4)}_{\text{amp}} =$  $=$  $+ \cdots$

$$= -i\lambda + \cdots. \tag{3.45}$$

3.3 Derivative Interactions

For the simple theories we have considered so far, all the results above can also be obtained in a straightforward manner using operator methods. However, the path integral approach is much simpler for more complicated theories. The classic example is gauge theories, which will be discussed later in the course. Another class of theories that are more easily treated using path integral methods are theories with derivative interactions. Consider for example a theory of a single scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2} g(\phi) \partial^\mu \phi \partial_\mu \phi - V(\phi), \tag{3.46}$$

where $g(\phi)$ is a given function of ϕ. This model is an example of what is called (for historical reasons) a **non-linear sigma model**.

Let us quantize this theory using the canonical formalism. The first step is to compute the conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = g(\phi) \dot{\phi}, \tag{3.47}$$

and the Hamiltonian density

$$\mathcal{H} = \frac{1}{2g(\phi)} \pi^2 + \frac{1}{2} g(\phi) \left(\vec{\nabla} \phi \right)^2 + V(\phi). \tag{3.48}$$

We see that the interacting part of the Hamiltonian depends on the conjugate momenta. When we work out the Feynman rules for this Hamiltonian using operator methods, we will find that the Wick contraction that defines the propagator is not relativistically covariant, and the interaction vertices are also not covariant. However, the underlying theory *is* covariant, and so the non-covariant pieces of the vertices and the propagator must cancel to give covariant results.

We can understand all this from the path integral. The path integral is analogous to the one for quantum mechanics with a position-dependent mass. We find that the path integral measure depends on the fields:

$$d[\phi] = C \prod_x g^{-1/2}(\phi_x) d\phi_x. \tag{3.49}$$

We can write the ϕ-dependent measure factor as a term in the action:

$$\Delta S = \frac{i}{2} \sum_x \ln g(\phi_x) \to \frac{i}{2} \frac{1}{a^4} \int d^4 x \, \ln g(\phi), \tag{3.50}$$

where a is a lattice spacing used to define the theory. This is highly divergent in the continuum limit, and we will be able to give it a proper treatment only after we have discussed renormalization. We will show then that this contribution can be consistently ignored.

If we trust that the extra measure factor can be ignored, the path integral quantization is very simple. The propagator is just the inverse of the kinetic term, and the vertices are read off by expanding in powers of ϕ around $\phi = 0$:

$$\mathcal{L} = \frac{1}{2}g(0)\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}g'(0)\phi\partial^\mu\phi\partial_\mu\phi + \cdots - \frac{1}{2}V''(0)\phi^2 - \frac{1}{3!}V'''(0)\phi^3 + \cdots \quad (3.51)$$

(We assume that $V'(0) = 0$, and we drop the irrelevant constant term $V(0)$.) Note that both the propagator and the vertices are manifestly Lorentz invariant.

Exercise: Derive the Feynman rules for this theory. Use them to calculate the tree-level contribution to the 4-point function $\langle\phi(x_1)\cdots\phi(x_4)\rangle$.

4 The Semiclassical Expansion

We now show that the perturbative expansion described above is closely related to an expansion around the classical limit. Our starting point is once again the path integral expression for the correlation functions:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \frac{\displaystyle\int d[\phi]\, e^{iS[\phi]/\hbar}\,\phi(x_1)\cdots\phi(x_n)}{\displaystyle\int d[\phi]\, e^{iS[\phi]/\hbar}}. \quad (4.1)$$

We have explicitly included the dependence on $\hbar$, since this will be useful in understanding the classical limit. (The classical action $S[\phi]$ is assumed not to depend on $\hbar$.)

Let us recall the intuitive understanding of the classical limit discussed briefly in the previous chapter. A given correlation function will be classical if the sum over paths is dominated by a classical path, that is, a field configuration $\phi_{\rm cl}(x)$ that makes the classical action stationary:

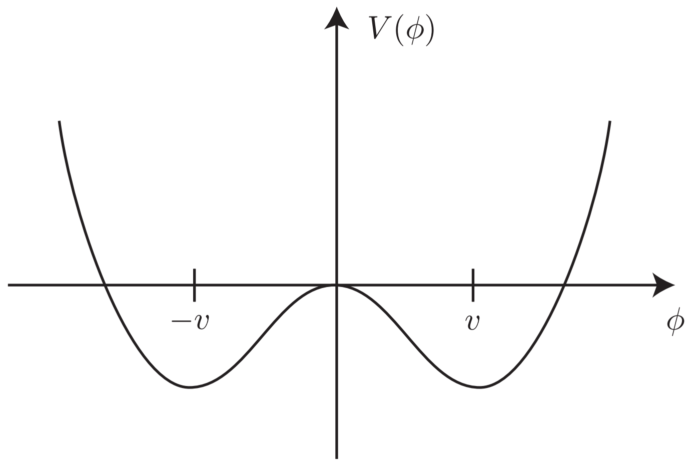
$$\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\rm cl}} = 0. \quad (4.2)$$

Intuitively, this is because paths close to the classical path interfere constructively, while other paths interfere destructively.

Before making this precise, let us give some examples of what kind of classical configurations we might be interested in. Consider our standby scalar field theory with a potential with a negative coefficient for the ϕ^2 term

$$V(\phi) = -\tfrac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4, \quad (4.3)$$

with $\mu^2 > 0$. This potential has the 'double well' form shown below:



It has three obvious constant classical solutions:

$$\phi_{\rm cl} = 0, \tag{4.4}$$

$$\phi_{\rm cl} = \pm v, \tag{4.5}$$

where

$$v = \left(\frac{6\mu^2}{\lambda} \right)^{1/2}. \tag{4.6}$$

The solution $\phi_{\rm cl} = 0$ is classically unstable, so we expect it to be unstable quantum-mechanically as well. The solutions $\phi_{\rm cl} = \pm v$ are the classical 'ground states' (lowest energy states), and are therefore the natural starting point for a semiclassical expansion. Note that these solutions are Lorentz invariant and translation invariant. However, they are not invariant under the symmetry $\phi \mapsto -\phi$.

In fact, this theory has other interesting classical solutions that we might use as the starting point for a semiclassical expansion. The classical equation of motion for time-independent field configurations is

$$\vec{\nabla}^2 \phi = \mu^2 \phi - \frac{\lambda}{3!} \phi^3. \tag{4.7}$$

We can consider a field configuration that depends only on a single spatial direction (z say) and satisfies the boundary condition

$$\phi(z) = \begin{cases} -v & \text{as } z \to -\infty \\ +v & \text{as } z \to +\infty \end{cases} \tag{4.8}$$

This has a simple solution

$$\phi_{\rm cl}(z) = v \tanh\left(\sqrt{2}\,\mu(z - z_0)\right), \tag{4.9}$$

where z_0 is a constant of integration. This is the lowest-energy state that satisfies the boundary condition Eq. (4.8). In this solution, the value of the field goes from $-v$ to $+v$ around the position $z = z_0$. This is called a **domain wall**.[3] It is the lowest energy configuration satisfying the boundary condition Eq. (4.8), and can therefore also be thought of as a classical ground state.

With these examples in mind, let us expand the path integral expression for the correlation function around a classical configuration $\phi_{\rm cl}$ that we think of as a classical ground state. We write the fields as

$$\phi = \phi_{\rm cl} + \phi', \tag{4.10}$$

ϕ' as parameterizes the quantum fluctuations about the solution $\phi_{\rm cl}$. We then compute correlation functions of the fluctuation fields We then expand the action about $\phi = \phi_{\rm cl}$:

$$S[\phi] = S[\phi_{\rm cl}] + \frac{1}{2!}\int d^4x_1 d^4x_2\; \frac{\delta^2 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{\phi=\phi_{\rm cl}} \phi'(x_1)\phi'(x_2) + \mathcal{O}(\phi'^3), \tag{4.11}$$

where we have used the fact that a linear term in ϕ' is absent by Eq. (4.2). We treat Eq. (4.10) as a change of variables in the path integral. Note that

$$d[\phi] = \prod_x d\phi(x) = \prod_x d\left[\phi'(x) + \phi_{\rm cl}(x)\right] = d[\phi'], \tag{4.12}$$

since the difference between ϕ' and ϕ is a constant (ϕ-independent) shift at each x. We treat the quadratic term as a free action in the exponential and expand the $\mathcal{O}(\phi'^3)$ and higher terms as 'interactions.' (The corresponding approximation for ordinary integrals is called the **stationary phase approximation**.) In this way, we obtain

$$\begin{aligned} Z[0] = e^{iS[\phi_{\rm cl}]/\hbar}\int d[\phi'] \exp\left\{\frac{i}{2!\hbar}\int d^4x_1 d^4x_2\; \frac{\delta^2 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{\phi=\phi_{\rm cl}} \phi'(x_1)\phi'(x_2)\right\} \\ \times\left[1 + \frac{i}{3!\hbar}\int d^4y_1\cdots d^4y_3\; \frac{\delta^3 S[\phi]}{\delta\phi(y_1)\cdots\delta\phi(y_3)}\bigg|_{\phi=\phi_{\rm cl}} \phi'(y_1)\cdots\phi'(y_3) + \cdots\right]. \end{aligned} \tag{4.13}$$

(Note that the $\delta^3 S/\delta\phi^3$ term may vanish because of a symmetry under $\phi \mapsto -\phi$, but higher order terms are nonvanishing unless the theory is trivial.) The numerator has a similar expansion, with additional powers of fields inside the path integral.

[3] In $1+1$ dimensions, this solution looks just like a particle, and is called a **soliton**.

This is in fact an expansion in powers of $\hbar$. To see this, define a rescaled field $\tilde{\phi}$ by

$$\phi' = \sqrt{\hbar}\tilde{\phi}. \tag{4.14}$$

In terms of this, Eq. (4.13) becomes

$$\begin{aligned} Z[0] = e^{iS[\phi_{\rm cl}]/\hbar} \int d[\tilde{\phi}] \exp\left\{ \frac{i}{2!} \int d^4x_1 d^4x_2 \; \frac{\delta^2 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{\phi=\phi_{\rm cl}} \tilde{\phi}(x_1)\tilde{\phi}(x_2) \right\} \\ \times \left[1 + \frac{i\hbar^{1/2}}{3!} \int d^4y_1 \cdots d^4y_3 \; \frac{\delta^3 S[\phi]}{\delta\phi(y_1)\cdots\delta\phi(y_3)}\bigg|_{\phi=\phi_{\rm cl}} \tilde{\phi}(y_1)\cdots\tilde{\phi}(y_3) + \mathcal{O}(\hbar^2) \right], \end{aligned} \tag{4.15}$$

where we have dropped an irrelevant overall constant from the change of measure. We now see that the leading contribution to Z is given by the exponential of the classical action. The $\mathcal{O}(\hbar)$ corrections from expanding the higher-order terms in the action vanish in the classical limit $\hbar \to 0$, and therefore parameterize the quantum corrections.

We can perform the same expansion on the numerator of Eq. (4.1). It is convenient to consider correlation functions of the fluctuation fields

$$\langle \phi'(x_1) \cdots \phi'(x_n) \rangle = \hbar^{n/2} \langle \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) \rangle, \tag{4.16}$$

and following the steps above we see that $\langle \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) \rangle$ has an expansion in powers of $\hbar$. This is called the **semiclassical expansion**.

For example, for the scalar field theory expanded about the solution $\phi_{\rm cl} = v$, the potential can be written in terms of the fluctuation fields as

$$V(\phi) = +\mu^2\phi'^2 + \frac{\lambda v}{3!}\phi'^3 + \frac{\lambda}{4!}\phi'^4. \tag{4.17}$$

Note in particular that the quadratic term is positive, so the fluctuations have a 'right-sign' mass. Therefore, the semiclassical expansion of this theory is equivalent to the ordinary diagrammatic expansion of the theory with field ϕ' with potential given by Eq. (4.17). Note that because the action is even in powers of ϕ', we have $\langle \phi' \rangle = 0$, and hence

$$\langle \phi \rangle = v. \tag{4.18}$$

Because $\langle \phi \rangle = \langle 0 | \hat{\phi}_H | 0 \rangle$, we say that the field ϕ has a nonzero **vacuum expectation value**. We will have much more to say about this later.

To see what terms contribute at a given order in the semiclassical expansion, it is convenient to go back to the Lagrangian in terms of the unshifted fields and define the rescaled fields

$$\phi = \sqrt{\hbar}\tilde{\phi}. \tag{4.19}$$

In terms of these, the action including a source term becomes

$$S/\hbar = \int d^4x \left[\frac{1}{2}\partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi} - \frac{m^2}{2}\tilde{\phi}^2 - \frac{\hbar\lambda}{4!}\tilde{\phi}^4 + \frac{1}{\sqrt{\hbar}} J\tilde{\phi} \right]. \tag{4.20}$$

From this, we see that the expansion of correlation functions in powers of λ is the same as the expansion in powers of $\hbar$, in the sense that

$$\begin{aligned} \langle \phi(x_1) \cdots \phi(x_n) \rangle &= \hbar^{n/2} \langle \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) \rangle \\ &= \hbar^{n/2} \times \text{function of } \hbar\lambda. \end{aligned} \tag{4.21}$$

This shows that successive terms in the weak-coupling expansion in powers of λ are suppressed by powers of $\hbar$. In this case, the dimensionless expansion parameter is $\hbar\lambda$.

The fact that the perturbative expansion is an expansion in the combination $\hbar\lambda$ in ϕ^4 theory depends on the form of the Lagrangian. However, we will now show that in any theory there is a close relation between the weak-coupling expansion and an expansion in powers of $\hbar$. Consider an arbitrary connected graph contributing to an n-point correlation function $\tilde{G}^{(n)}$ in an arbitrary field theory (in any dimension). Let us count the number of powers of $\hbar$ in this graph. The propagator is the inverse of the quadratic term in the action, and is therefore proportional to $\hbar$. Each vertex is proportional to a term in the action, and is therefore proportional to $1/\hbar$. Therefore,

$$\text{graph} \sim \hbar^{n+I-V}, \tag{4.22}$$

where n is the number of external lines, I is the number of internal lines, and V is the number of vertices. But for any connected graph,

$$L = I - V + 1, \tag{4.23}$$

where L is the number of loops (independent momentum integrals) in the diagram. The reason for this is that each propagator has a momentum integral, but each vertex has a momentum-conserving delta function that reduces the number of independent momentum integrals by one. There is an additional factor of $+1$ from the fact that one of the momentum-conserving delta functions corresponds to overall momentum conservation, and therefore does not reduce the number of momentum integrations. Combining Eqs. (4.22) and (4.23), we obtain

$$\text{graph} \sim \hbar^{n-1+L}. \tag{4.24}$$

Therefore graphs with additional loops are suppressed by additional powers of $\hbar$. This shows that the loop graphs can be viewed as quantum corrections.

5 Relation to Statistical Mechanics

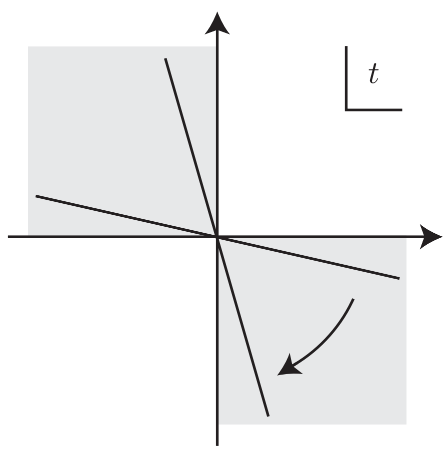
We have seen that in quantum field theory, the basic object of study is the generating functional

$$Z[J] = \int d[\phi] e^{i\left(S[\phi] + \int J\phi\right)}. \tag{5.1}$$

The $i\epsilon$ prescription is crucial to make the integral well-defined. This can be viewed as an infinitesimal rotation of the integration contour in the complex t plane:

$$x^0 = (1 - i\epsilon)\tau, \qquad \tau = \text{real}. \tag{5.2}$$

We can continue rotating the contour to purely imaginary t if there are no singularities in the second and fourth quadrants of the complex t plane:



We will later show that there are no singularities to obstruct this continuation (at least to all orders in perturbation theory), so we can go all the way to the imaginary time axis:

$$x^0 = -i x^0_{\text{E}}, \qquad x^0_{\text{E}} = \text{real}. \tag{5.3}$$

Then

$$d^4x = -id^4x_{\rm E}, \quad \frac{\partial}{\partial x^0} = i\frac{\partial}{\partial x^0_{\rm E}}, \tag{5.4}$$

etc. This gives

$$\partial^\mu\phi\partial_\mu\phi = (\partial_0\phi)^2 - (\partial_i\phi)^2 = -(\partial_{\rm E0}\phi)^2 - (\partial_i\phi)^2 \stackrel{\rm def}{=} -(\partial_{\rm E}\phi)^2. \tag{5.5}$$

The metric has become (minus) a Euclidean metric. The action for ϕ^4 theory can then be written

$$\begin{aligned} iS &= \int\Big(-id^4x_{\rm E}\Big)\left(-\frac{1}{2}(\partial_{\rm E}\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right) \\ &= -S_{\rm E}, \end{aligned} \tag{5.6}$$

where

$$S_{\rm E} = \int d^4x_{\rm E}\left[\frac{1}{2}(\partial_{\rm E}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right] \tag{5.7}$$

is the **Euclidean action**. Defining

$$J_{\rm E}/\hbar \stackrel{\rm def}{=} iJ \tag{5.8}$$

we have

$$Z_{\rm E}[J_{\rm E}] = Z[J] = \int d[\phi]\, \exp\left\{-\frac{1}{\hbar}\left(S_{\rm E}[\phi] + \int J_{\rm E}\,\phi\right)\right\}, \tag{5.9}$$

where we have explicitly included the factor of $\hbar$.

We note that Eq. (5.9) has exactly the form of a partition function for classical statistical mechanics:

$$Z_{\rm stat\,mech} = \sum_{\rm states} e^{-H/T}, \tag{5.10}$$

where H is the Hamiltonian and T is the temperature. In the analog statistical mechanics system, we can think of $\phi(x)$ as a 'spin' variable living on a space of 4 *spatial* dimensions. The Hamiltonian of the statistical mechanics system is identified with the Euclidean action $S_{\rm E}$. Note that $S_{\rm E}$ is positive-definite, so the energy in the statistical mechanics model is bounded from below. The source term $\int J_{\rm E}\,\phi$ in the Euclidean action corresponds to a coupling of an external 'magnetic' field to the spins. Finally, $\hbar$ plays the role of the 'temperature' of the system. This makes sense, because quantum fluctuations are large if $\hbar$ is large (compared to other relevant scales in the

Quantum Field Theory	Classical Statistical Mechanics
$3+1$ spacetime dimensions	4 spatial dimensions
$\phi(x)$	spin variable
$J(x)$	external field
$S_{\rm E}$	Hamiltonian
$\hbar$	temperature
$\langle\phi(x)\rangle$	magnetization

Table 1. Relation between quantities in quantum field theory and classical statistical mechanics.

problem), while thermal fluctuations are large for large T. This is very precise and deep correspondence, which means that many ideas and techniques from statistical mechanics are directly applicable to quantum field theory (and *vice versa*).

One very important connection between the two subjects is in the subject of phase transitions. It is well-known that statistical mechanical systems can undergo phase transitions as we vary the temperature, external fields, or other control parameters. The occurrence of a phase transition is usually signaled by the the value of certain order parameters. In a spin system, the simplest order parameter is the magnetization, the expectation value of a single spin variable, which is nonzero only in the 'magnetized' state. In the quantum field theory, the analog of the magnetization is the 1-point function $\langle\phi_x\rangle$. Note that in the ϕ^4 theory, the Lagrangian is invariant under $\phi\mapsto-\phi$, and a nonzero value for $\langle\phi_x\rangle$ signals a breakdown of this symmetry. For example, in the previous section, we argued that the semiclassical expansion suggests that the field ϕ has a nonzero vacuum expectation value when the coefficient of the ϕ^2 term in the Lagrangian is negative. This important subject will be discussed in more detail later.

6 Convergence of the Perturbative Expansion

We now discuss briefly the convergence of the weak coupling perturbative expansion described above. A simple physical argument shows that the series cannot possibly converge on physical grounds. Consider the ϕ^4 theory that we have been using as an example. If the expansion in powers of λ converged, it would define an analytic function of complex λ in a finite region around $\lambda = 0$. But this would mean that the expansion converged for some negative values of λ. Physically, this cannot be, since the theory for $\lambda < 0$ has a potential that is unbounded from below, and the theory should not make sense.[4]

This argument was originally given by Freeman Dyson for quantum electrodynamics. In that case, the weak coupling expansion is an expansion in powers of $\alpha = e^2/4\pi$. If it converged for some finite $\alpha > 0$, it would have to converge for some $\alpha < 0$. However, in this theory we can lower the energy of the vacuum state by adding e^+e^- pairs to the vacuum, so we expect an instability in this case as well.

Although the perturbative series does not converge, we do expect that the interacting theory reduces to the free theory in the limit $\lambda \to 0+$ (*i.e.* we take the limit from the positive direction). Now consider a truncation of the series containing only the terms up to $\mathcal{O}(\lambda^n)$ for some fixed n. Clearly there is a sufficiently small value of λ such that the successive terms in the series get monotonically smaller. For these small values of λ, we expect that keeping more and more terms in the truncation will make the approximation more accurate. However, if we include higher and higher terms in the series, it must eventually diverge. A series with this property is called an **asymptotic series**. For any value of λ, there is an optimal truncation of the series that gives the best accuracy.

These properties can be seen in a simple model of '0 + 0 dimensional Euclidean field theory,' *i.e.* the ordinary integral

$$Z = \int_{-\infty}^{\infty} d\phi \, \exp\left\{ -\frac{1}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right\}. \tag{6.1}$$

The expansion in powers of λ diverges because the integral becomes ill-defined for $\lambda < 0$. We can expand this in powers of λ

$$Z = \sum_{n=0}^{\infty} Z_n \lambda^n, \tag{6.2}$$

with

$$Z_n = \frac{2}{n!} \left(-\frac{1}{4!} \right)^n \int_0^{\infty} d\phi \, \phi^{4n} e^{-\phi^2/2}. \tag{6.3}$$

We can find the asymptotic behavior of Z_n for large n using the method of steepest descents. We write the integrand as $e^{-f(\phi)}$, with

$$f(\phi) = \tfrac{1}{2}\phi^2 - 4n \ln \phi. \tag{6.4}$$

This has a minimum at $\phi^2 = 4n$. Expanding about this minimum, we have

$$f(\phi) = 2n(1 - \ln 4n) + (\phi - 2\sqrt{n})^2 + \cdots. \tag{6.5}$$

[4] As we have seen, the semiclassical expansion is also a weak coupling expansion, and the same arguments apply to it.

We therefore obtain

$$
\begin{aligned}
Z_n &\simeq \frac{2}{n!}\left(-\frac{1}{4!}\right)^n e^{-2n(1-\ln 4n)} \int_{-\infty}^{\infty} d\phi\, e^{-(\phi-2\sqrt{n})^2} \\
&= \frac{\sqrt{\pi}}{n!}\left(-\frac{1}{4!\,e^2}\right)^n e^{2n\ln 4n}.
\end{aligned}
\tag{6.6}
$$

Using the Stirling formula $n! \simeq e^{n\ln n - n}$, we see that for large n the coefficients in the series behave as

$$
Z_n \simeq \sqrt{\pi}\left(-\frac{1}{4!\,e}\right)^n e^{2n\ln 4n - n\ln n}. \tag{6.7}
$$

We can check when the successive terms in the series begin to diverge by computing the ratio

$$
\frac{Z_{n+1}\lambda^{n+1}}{Z_n\lambda^n} \simeq \frac{\lambda}{4!}4n. \tag{6.8}
$$

A good guess is that the optimal number of terms is such that this ratio is of order 1, which gives

$$
n_{\rm opt} \simeq \frac{3!}{\lambda}. \tag{6.9}
$$

For example, if $\lambda \simeq 0.1$, we must go out to about 60 terms in the series before it starts to diverge! We can estimate the error by the size of the last term kept, which gives

$$
\delta Z \sim \sqrt{\pi}e^{-n_{\rm opt}} \sim Z_0 e^{-3!/\lambda}. \tag{6.10}
$$

This is the typical behavior expected in asymptotic series. Note that the error is smaller than any power of λ for small λ.