TORSIONAL MODE OSCILLATIONS AND THE NEUTRON STAR EQUATION OF STATE

STEPHEN R. SCHWARTZ
Department of Astronomy, University of Maryland, College Park, Maryland 20742-2421
srs@astro.umd.edu

ABSTRACT

Neutron stars were theorized as an explanation for the copious energy yield of supernovae explosions long before the stars were ever observed. Had there not already been a great deal of time and effort devoted to theorizing the properties of these objects by the time they were discovered, we would have been completely dumbfounded by our observations. As it was, the astrophysics community was only somewhat dumbfounded at what was eventually determined to be a radio pulsar. Forty years later, a great deal of the physics that govern neutron stars have been uncovered, although the vast majority remains to be probed. Recent observations of magnetar giant flares show quasi-periodic oscillations (QPOs) left in their wake. It is likely that most of these QPOs can be associated with elastic toroidal torsional mode oscillations taking place within the crust of the neutron star itself as a result of magnetic field line reconfiguration within the crust, believed to be the cause of the flare. What can these flares tell us about the structure of the neutron star? And why is it the case that all low l fundamental toroidal torsional modes do not show excitation resulting from these flares?
Subject headings: stars: neutron — stars: magnetic fields—stars: oscillations

1. INTRODUCTION

It could certainly be argued that an accurate and detailed model representing the neutron star equation of state would prove to be the holy grail for many fields of physics and astrophysics. Its discovery would be a boon for particle physics, big-bang theory, gravitational physics, high-energy-astrophysics, quantum and nuclear physics, and theoretical solid-state physics. Although in the end the equation of state must be a theoretical model, and can therefore never truly be said to be *found*, to approach it requires the organic interplay between observation, experimentation, and theory.

In the earlier half of the twentieth century, Walter Baade and Fritz Zwicky were looking for the mechanism behind supernovae explosions. Once the neutron was discovered in 1932, within about a year of the discovery, they postulated the existence of the neutron star: an extremely dense object comprised of about a solar mass of material, supported against gravity by neutron degeneracy pressure. The gravitational binding energy of such an object would be more than enough to account for the the energy released in a supernova. However, it was not for over thirty years after its prediction, that we finally gained clear observational evidence for such an extremely compact object. In 1967, a rapidly pulsating radio source was first observed. Since the idea of neutron stars was known among much of the astrophysics community at the time, it took under a year to determine that this source was indeed the *very first* instance of the *very first* discovered class of neutron star, a radio pulsar, PSR 1919+21.

It seems that observation, experimentation, and theory are often obliged to wait for one another to catch up before they each can progress. When observation caught up with and passed neutron star theory in one discrete jump with the incidental discovery of the first pulsar, the theoretical horizons in the area of compact objects once again opened up. Since then, each important new set of neutron star observations has begged for explanation. And each resultant idea has given us the impetus and direction to seek out the next breakthrough observation. As the constraints on the properties of neutron stars grow, so do the details of the theories and the models that describe them.

More recently, both the observations of neutron stars in accreting binaries and observations of quasi-periodic oscillations (QPOs) in the wake of magnetar giant flares have inspired the formulation of great theoretical and computational tools with which to plumb the depths of neutron stars. This paper is concerned with the discovery of neutron stars, two major classes of neutron star, namely, the pulsar and the magnetar, and the interpretation of the QPOs associated with oscillations in the solid crusts of magnetars, and how they help to shed light on the neutron star equation of state.

2. CLASSES OF NEUTRON STARS

a) Pulsars

In 1967, a young graduate student at Cambridge University by the name of Jocelyn Bell discovered a radio source that emitted a pulse extremely regularly: once every 4/3 seconds (Bell & Hewish 1967). What kind of a star could rotate so quickly? Before concluding that the source was indeed a star, Bell's adviser, Tony Hewish, who would later receive the Nobel Prize for this discovery, took seriously the possibility that an extra-solar civilization may be intentionally responsible for the short, precise periodicity of the pulses. Once terrestrial (earth-bound) sources were ruled out, there were very few explanations left. If we make a rough estimate of what the very maximum size of the source could be if it were to indeed be stellar, we come up with

$$\frac{2\pi R}{v_{esc}} < \frac{4}{3}\ s \ \rightarrow\ R < \frac{4 v_{esc}}{6\pi}\ ,$$

where $v_{esc} = \sqrt{2GM/R}$.

$$\text{This gives } R \lesssim 2000\ km \left(\frac{M}{M_{SUN}}\right)^{1/3} . \qquad (1)$$

A white dwarf, even in an extreme case of both being right near the Chandrasekhar limit of ~$1.4M_{\mathrm{SUN}}$ and rotating right near its breakup speed, would be difficult to fit into the density criterion estimated in (1).[1] However this might turn out, the explanation was bound to be an exotic one.

Bell spent several more weeks pouring over the data, which consisted of literally miles of paper that had been scrawled on by the pen recorders that she used. After she found at least three additional pulsating radio sources, Hewish and company reasoned that simultaneous deliberate signals, sent by multiple sentient species around the galaxy directly toward Earth, all using the same means of contact, did not sound like the most reasonable explanation. More likely, these were proof of the existence of the the extreme-density neutron degenerate stars postulated by Zwicky and Baade, with sidereal days on the order of one second. In fact, had Zwicky and Baade not already theorized the existence of the neutron star over thirty years earlier, perhaps we would have had a short era where a large fraction of the scientific community believed that we were intentionally being contacted by intelligent life outside of our solar system.

[1]Pulsars have since been observed, for instance by Hessel *et al.* (2006), with frequencies over a thousand fold greater than that of PSR 1919+21.

Instead, the pulses were due to synchrotron radiation from electrons spinning around the strong magnetic field lines of the compacted stars. The power radiated away by synchrotron radiation is given by the relativistic Larmor formula (Larmor was indeed an earthling):

$$P = \frac{2e^2}{3c^3} \vec{a}^2 \gamma^4 , \quad (2)$$

where e is the charge of the electron, $\vec{a}$ is the acceleration of the electron moving across the magnetic field lines, and γ is the relativistic Lorentz factor. The rotational axis need not be aligned with the magnetic axis. In this case there will be a radiation beam sweeping very regularly, with a period equal to the spin rate of the neutron star, across the portions of the sky in line with the magnetic axis.

If the Earth happens to be within the portion of sky that is swept out by the neutron star's radiation field, it is then possible that we will detect it as a pulsar. So we call a pulsar those neutron stars whose north or south magnetic poles we can peer down once per revolution, as we are exposed to its radiation. Therefore, all pulsars are neutron stars, but all neutron stars need not be pulsars. In fact, the pulsar is only one sub-class of neutron star—another is the magnetar.

b) Magnetars

The magnetar is a class of neutron star with an exceptionally strong magnetic field. They are typically young and slow-rotating, exhibiting magnetic field strength several orders of magnitude greater than any object in the known universe, which can be upward of 10^{15} gauss (Woods, *et al.* 2004). A magnetar has a very rapid spin-down rate due to its magnetic coupling to ions far from its surface—a mechanism that can cause a star to lose a copious amount of angular momentum over a relatively short period of time. The structure of the magnetar is theorized to be similar to that of other neutron stars, other than the effects borne of its ultra-strong magnetic field.

3. NEUTRON STAR AND MAGNETAR STRUCTURE

The very surface of the neutron star is assumed to be covered in liquid iron peak elements near ^{56}Fe. The depth of this 'ocean' is thought to be on the order of 10 meters. This liquid metal ocean rests upon a crust of solid nuclei. This solid portion of the star extends down through some fraction of the star's radius, which my results show to be just under ten percent. It doesn't take long for densities in the crust to exceed the neutron drip point (which is taken to be at $\rho_{drip} \sim 4.3 \times 10^{11}$ g/cm^3). This marks the the boundary between crust and mantle (sometimes upper (or outer) crust and lower (or inner) crust). At this density, neutrons begin to become superfluid, and very exotic nuclei are posited to exist under these extreme conditions, these include extremely heavy nuclei in the "Pasta Regime" (Carter *et al.* 2005). It was Nuclei were believed to get as heavy as $A \sim 1000$ near the the base of the mantle. As we attempt to probe further, things get even more exotic and we come up against the bounds of our current knowledge. The inner structure of the neutron star, where density and pressure are highest, is a topic of heated debate within the field of high-energy astrophysics. Some think that the superfluid neutrons stay intact all the way down to the center of the star, others postulate a sea of quark matter, entailing the theory that neutron degeneracy pressure can be overcome in matter before it reaches black hole density. With few constraints, there is room for

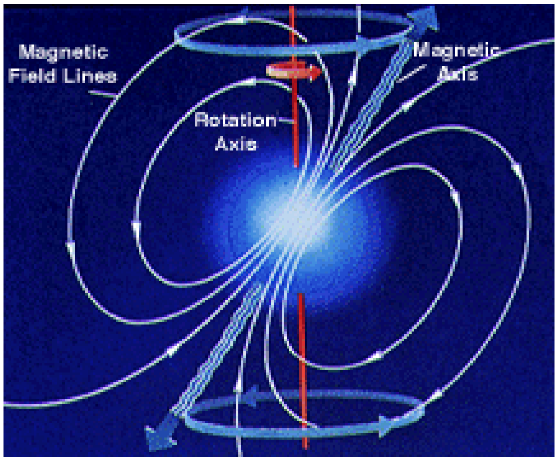

Fig. 1: *A diagram of a pulsar: shown are its rotation axis and its magnetic axis.*

http://imagine.gsfc.nasa.gov/

the imagination to flow. And indeed there exist a multitude of theories that attempt to explain the central regions of neutron stars.

But now back to the solid portion of the star: the crust and mantle take up only a small percentage of the star's total mass, but a significant fraction of the total volume. And since we have some capability to exploit the properties of a solid to constrain the equation of state, it plays a crucial role in our understanding of neutron stars.

QPOs have been observed following the giant flares of three soft γ-ray repeaters (SGRs), taken to be magnetars: SGR 0526-66: [Barat *et al.* (1983)]; SGR 1806-20 [Israel *et al.* (2005), Strohmayer & Watts (2006), Watts & Reddy (2006), Watts & Strohmayer (2006a), and Watts & Strohmayer (2006b)]; SGR 1900+14 [Strohmayer & Watts (2005) and Watts & Reddy (2006)]. These giant flares are thought to result from the repositioning of magnetic field lines within the crust of magnetars. The ions and electrons in these stars are coupled to the stars' magnetic field. Hence any reconfiguration of the field may cause large-scale fractures and displacements in the crust. Restoring forces are brought about by the resulting displacement from the nearest locally stable crustal configuration. These displacements (which can in principle vary widely from one flare to the next) along with the equilibrium states of the star, determine the types of oscillations[2] that result from such a given event.[3]

Beyond the rare and difficult detection of these flares, there is the task of recognizing the periodicity in the observed lightcurve as the flare trails off. Then, once there is confidence in the recognition of these periodicities in the observed lightcurve, there is the daunting task of matching up these perceived oscillations in the tails of these flares with appropriate models of a magnetar crust capable to exhibit these modes of oscillation. The more confidence we have in the models, and the more detailed the models become, the more we have constrained the equation of state of the star.

Magnetars have, on the three occasions named, been able to produce displacements great enough for the detection of the resulting modulation in the X-ray spectrum. Quasi-periodic oscillations have been detected in SGR 1806-20 by Israel *et al.* (2005) at 18.1 Hz $\pm$0.3, 30 Hz $\pm$0.3, 92.5 Hz, and in SGR 1900+14 by Strohmayer & Watts (2005) at 28 Hz $\pm$0.5, 53.5 Hz $\pm$0.5, 84 Hz, 155.1 Hz $\pm$0.2. The toroidal and shear modes are used to model the detected signals because they are efficient in

---

2 When thinking about torsional modes, it may be helpful to imagine the neutron star as a complex spherical system of springs, hollow in the center, with flexibility only in non-radial directions. If the sphere is compressed or distorted in some (nonspherically symmetric) fashion, assuming an underdamped system, torsional oscillations are inevitable.

3 The magnetic field lines of neutron stars with more moderate field strength (10^{12} – 10^{13} G) may be able to reconfigure themselves without global fractures of the (more malleable) crust. Alternatively, it could be the case that starquakes are common in the crusts of most classes of neutron star, but the energy release is too small to have been detected and/or properly identified.

modulating the X-ray lightcurve for periodicity (Blaes *et al.* 1989). Besides the low frequency 18.1 Hz oscillation, which could be an internal mode restored after the flare by the strong field of the magnetar (Israel *et al.*, 2005), all of the QPOs listed above seem to fit with fundamental toroidal oscillation modes. More recently, a 150 Hz QPO, a 625 Hz QPO, and a 1.84 kHz QPO were detected in the more recent flare from SGR 1806-20 (Watts & Strohmayer 2006b). The latter two may represent higher order toroidal torsional modes, especially the 625 Hz QPO, which fits very well with models of the n=1 mode, and has been detected by both the *Rossi X-ray Timing Explorer* (RXTE) and the *Ramaty High Energy Solar Spectroscopic Imager* (RHESSI).

4. MODELING THE OSCILLATIONS

a) Rotation

Rotation has the effect of coupling the toroidal modes, the modes that I'll pay most attention to here, with the spheroidal modes. Since everything rotates, adding in the effects of rotation should always make the model more accurate. However, in the case of modeling even the lowest frequency toroidal modes in a magnetar, the coupling turns out to be insignificant (Lee 2007). For significant coupling to occur, the rotational frequency must be on the same order as the oscillation frequencies of the modes being examined. Unlike typical pulsars, magnetars are slow rotators, with frequencies well under 1 Hz, while models of the lowest frequency torroidal mode, the ${}_2t_0$ mode, show that typical frequencies are usually at least two orders of magnitude higher, around 20-30 Hz, depending on the model.

b) The Magnetic Field

Magnetars, despite the name having been derived from being the objects which exhibit the greatest magnetic field strength observed in the universe, a few times 10^{15} gauss (Woods, *et al.* 2004), do not have magnetic fields strong enough to influence their most common toroidal torsional oscillation modes. The fundamental modes are hardly influenced by the magnetic fields that are exhibited by even the most magnetized neutron stars (also shown by Piro (2005) for $B \sim 4\text{x}10^{15}$ gauss, and Lee (2007) for $B \sim 10^{15}$ gauss). However, for the higher radial order terms, $n > 0$, magnetic field effects tend to add to the frequencies, which increase with increasing n. Also, the ${}_lt_n$ modes, already at indistinguishable frequencies for low l, tend to be coupled together in the presence of the strong fields exhibited by magnetars for a given value of n, especially at these lower values of l (Piro 2005, Lee 2007).

c) The Cowling Approximation

Metric perturbations can be ignored while still preserving accuracy since the toroidal modes that are under consideration are confined to the crust, which lies within a region of low enough density that the perturbations to the spacetime metric are sufficiently minute, and reasonable to neglect (Samuelsson & Andersson, 2007). In fact, I put in a general relativistic correction term for the gravitational potential to solve for a modified Lane-Emden equation. The result was that there was no significant difference to the density distribution outside the very inner core of the model star.

d) The Shear Modulus

The behavior of the oscillations, including their frequencies, can be quite accurately predicted by taking the approximation that the shear modulus, μ, depends linearly on the density in the crust. The exact behavior of the shear modulus within the crust, including its derivative with respect to crustal depth, $d\mu/dr$, are poorly understood quantitatively. However, according to Ruderman (1968), McDermott *et al.* (1988), and Piro (2005), the shear speed, equal to $(\mu/\rho)^{1/2}$, is nearly constant at 10^8 cm/s (see v_s in Figure 2, Piro, 2005).

As long as this approximation is used, the equation will be greatly simplified because μ is taken to be directly proportional to ρ, as is $d\mu/dr$ to $d\rho/dr$. It may be useful to compare this approximation to different models to describe the behavior of the shear modulus.

A more robust model would involve using a polytropic model, or integrating the full Tollman - Oppenheimer – Volkov (TOV) equation. I would also like to test the accuracy of using the constant shear speed approximation by attempting to more precisely solve for the shear modulus, given by, for example, Pandharipande *et al.* (1976) or Strohmayer *et al.*, (1991).

5. CONCLUSIONS ON THE CRUSTAL OSCILLATIONS

Matching the observed QPO frequencies to modes of oscillation is a very difficult task. There exist many free parameters that have to be taken into consideration, and knowledge of the oscillation frequencies alone is not enough to fully constrain them. The frequencies that we observe could belong to multiple modes of oscillation along with the toroidal torsional modes. That being said, there still exists great potential to constrain our equation of state models of neutron stars with the observations of the tails of these magnetar giant flares. Limiting the speculation to the lower frequencies for the moment, it seems likely that these are indeed caused by the low l fundamental modes with $n = 0$. It seems consistent with the data that the ${}_2t_0$ mode is excited in both the hyperflares of SGR1900+14 and SGE 1806-20. This makes intuitive sense as well because it is a fairly simple oscillation configuration with the lowest toroidal oscillation frequency.

But it seems to be a fundamental gap in our understanding that the observed frequencies do not occur for each value of l at $n = 0$, but only for intermittent values of l. The explanation may lie within the nature of the explosion rather than just the structure of the star. Thus far, this problem has been approached by solving the perturbation equations using our best guesses at the interior of the magnetar. It might be necessary to attempt to put more constraints on the types of crustal displacements that can occur as a result of these flares. Perhaps there may be a symmetry in the starquake that we have not taken into consideration, which could account for why, say, only the first even values of the fundamental modes seem to be excited, but not the odd values. It might be due to the complex interaction of the crustal deformities caused by the field lines' reconfiguration. Imagine that the flare displaces the crust in a finite number of positions with varying magnitudes. It should make sense that the interaction between all deformities made to the crust during a giant flare could either be constructive or destructive for each mode of oscillation during their return to equilibrium. For instance, it is believed that the actual physical fracture of the crust may stretch on the order of one to five kilometers across the surface (Schwartz *et al.* 2005). The shapes and sizes of the fractures will affect the excitation of specific modes in the solid

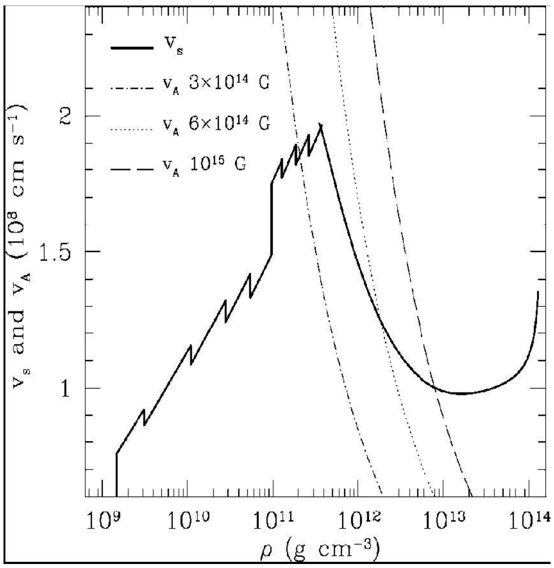

Fig. 2: *The shear speed is believed to stay roughly constant throught the neutron star's crust.*

Piro (2005)

portion of the star. After acquiring enough data, and enough confidence in both the data and in our models, the next step might be to attempt to model the actual explosions on a simplified model of the star's crust by inferring where and with what magnitude displacements occur in the crust. We do this while keeping in mind what types of displacements could conceivably be brought about by the mechanism of these types of explosions.

By delicately choosing which parameters to focus on and which ones to deem less relevant, we are evolving tentative, skeptical hypotheses about the interiors of neutron stars. With the inevitable growth of data obtained in order to test these hypotheses, these hypotheses can only be improved upon.

Though I suppose we have come quite far since Jocelyn Bell first noticed those radio pulses in 1967, there is still a very long way to go in understanding the properties of neutron stars. I believe that mapping out these oscillations can give us great insight into the equation of state. If we can increase our sample size (beyond a mere two of these events!), and are able to determine with confidence which types of oscillations and which modes become excited, we can learn a great deal about the interior of these stars.

It is certainly possible to constrain the type of matter in the core if we are indeed ever able to develop a detailed map of these QPOs, discovering how the oscillations propagate through the core. Besides particle physics, a detailed model of the interiors of neutron stars would provide clear tests of general relativity right at the point where it meets quantum mechanics—which is what most of us are really waiting for.

APPENDIX

In order to model the toroidal torsional modes of the crust, we start with the three governing equations of motion for a mass element: the momentum equation, the continuity equation, and Poisson's equation:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = \frac{1}{\rho} \nabla \cdot \sigma - \nabla \Phi \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{4}$$

$$\nabla^2 \Phi = 4\pi G \rho \tag{5}$$

Within the crust, the shear modulus is given by Strohmayer *et al.* (1991):

$$\mu = \frac{0.1194}{1+0.595(173/\Gamma)} \frac{n_i (Ze)^2}{a}. \tag{6}$$

The shear modulus describes the stress tensor for toroidal displacements:

$$\delta\sigma_{ij} = \mu\left(\frac{\partial\xi_i}{\partial x_j} + \frac{\partial\xi_j}{\partial x_i}\right) \tag{7}$$

Next we assume an oscillatory time dependence where σ (different from the shear stress tensor σ_{ij}) is the frequency of the oscillation mode in the neutron star. We now have (McDermott *et al.* 1988, Piro 2005):

$$\rho W\sigma^2 = -\frac{d\mu}{dr}\left(\frac{dW}{dr} - \frac{W}{r}\right) - \frac{\mu}{r^2}\frac{d}{dr}\left(r^2\frac{dW}{dr}\right) + \frac{l(l+1)}{r^2}\mu W \tag{8}$$

Now I will approximate $(\mu/\rho)^{1/2} \sim 10^{16}$ and introduce the following:

$$S \equiv \frac{W}{r}, \quad a \equiv l(l+1)-2, \quad q \equiv \frac{dS}{dr}, \quad b \equiv \sigma^2 x\, 10^{16}. \tag{9}$$

The following is the equation of motion within the crust, where S is a dimensionless quantity, proportional to [1/r] times the displacement amplitude.

$$\frac{dq}{dr} = S\left(\frac{a}{r^2} - b\right) - q\left(\frac{(du/dr)}{\mu} + \frac{4}{r}\right) \tag{10}$$

Any values of l and σ which allow q to be zero at both interfaces of the of the crust represent supported toroidal torsional oscillation modes within the crust and their corresponding frequencies.

REFERENCES

Barat, C. *et al.* 1983, Astron. Astrophys., 126, 400.
Bell, S.J. And Hewish, A. 1967, Nature, 213, 1214B.
Blaes, O. *et al.* 1989, ApJ, 343, 839.
Carter, B. *et al.* 2005, Nuc. Phys. A.748, 675C.
Hessel, J.W.T., *et al.* 2006, astro-ph/0601337v1.
Israel G.L., *et al.* 2005, ApJ, 628, L53.
Lee, U. 2007, Mon. Not. R. Astron. Soc., 374, 1015.
McDermott, P.N. *et al.* 1988, ApJ, 325, 725.
Pandharipande, V.R., Pines, D., and Smith, R.A. 1976, ApJ, 208, 550.
Piro, A.L. 2005, ApJ, 634, L153.
Ruderman, M.A. 1968, Nature, 218, 1128.
Samuelsson, L. and Andersson, N. 2007, Mon. Not. R. Astron. Soc., 374, 256.
Schwartz, S.J., *et al.* 2005, ApJ 627, L129.
Strohmayer, T.E., *et al.* 1991, ApJ, 375, 679.
Strohmayer, T.E. and Watts, A. L. 2005, Bull. of Am. Astron. Soc., 37, 1497.
Watts, A. L. and Reddy, S., 2006, astro-ph/0609364.
Watts, A. L. and Strohmayer, T. E., 2006a, ApJ, 637, L117.
Watts, A. L. and Strohmayer, T. E., 2006b, ApJ, 653, 593.
Woods, P.M. and Thompson, C. 2004, *Compact Stellar Xray Sources*, eds. Lewin, W.H.G., and van der Klis, M., Cambridge University Press, astro-ph/0406133.