Intro to GR

Einstein's motivations:

1) Newtonian instantaneous – conflicts w/ SR
2) Mach's principle : inertia <u>determined</u> by matter
   turned out overstated, but <u>spirit</u> correct :
   inertia <u>dynamical</u>, <u>influenced</u> by matter
3) Gravity pseudo force – equivalence principle.

+ 8 years of struggle & serious mathematical help.
→ GR.

(1)

$$\vec{F}_{grav} = m_{inertial}\,\vec{a}$$

$$\vec{F}_{grav} = m_{\substack{grav,\\ passive}}\,\vec{g} = -m\vec{\nabla}\phi$$

$$\nabla^2\phi = 4\pi G \rho_{\substack{grav\\ active}}$$

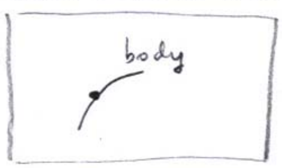
obs: all masses equiv.

$$\vec{a} = \frac{\vec{F}_{grav}}{m_{inertial}} = \underbrace{\frac{m_{grav,p}}{m_{inertial}}}_{=1}\,\vec{g}$$

So can remove grav by going to an "accelerated frame"

(2)

accelerated frame



$\vec{a}_{frame}$ →

$$\vec{a}_{body} = \vec{a}_{frame} + \vec{a}_{b\text{-}f}$$

$$\vec{F}_{ext} = m\vec{a}_{body} = m\vec{a}_{frame} + m\vec{a}_{bf}$$

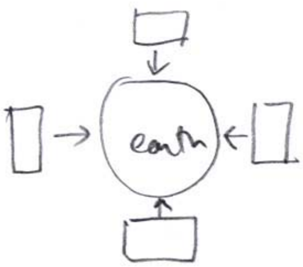
$$\to \quad \vec{F}_{ext} \underbrace{-\,m\vec{a}_{frame}}_{} = m\vec{a}_{bf}$$

pseudo-force, $\propto m$, eliminate by using <u>inertial</u> frame.

So grav ⇔ pseudo-force, $\vec{F}_{grav} = -m\vec{a}_{frame}$.

③ But... gravity "real", no?

e.g.



local inertial frames don't fit together into one global frame.

since $\vec{\nabla} \vec{g} \neq 0$.

This is true gravity.

→ free-fall inertial motion wrt. local inertial frames, gravity variation of these frames. How to describe this?

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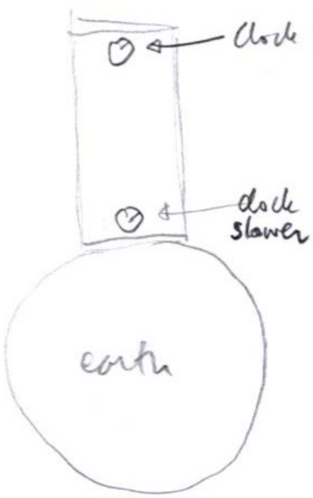
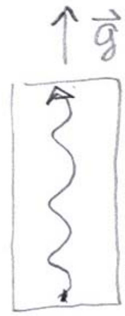
(4) A clue: effect of gravity on time





in small enough region

interval between two signals received longer than sent.

$t_2 - t_1 = t + \varepsilon$

$t_2 = t + \varepsilon + t_1$

NR:

$d + \frac{1}{2} a t_1^2 = t_1 \to t_1 = d$.

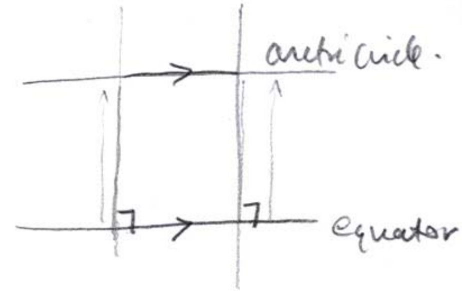
$d + \frac{1}{2} a t_2^2 - \frac{1}{2} a t^2 = t_2 - t$

$t_2 - t_1 = \frac{1}{2} a (t_2 - t_1)(t_2 + t_1) - \frac{1}{2} a t^2$

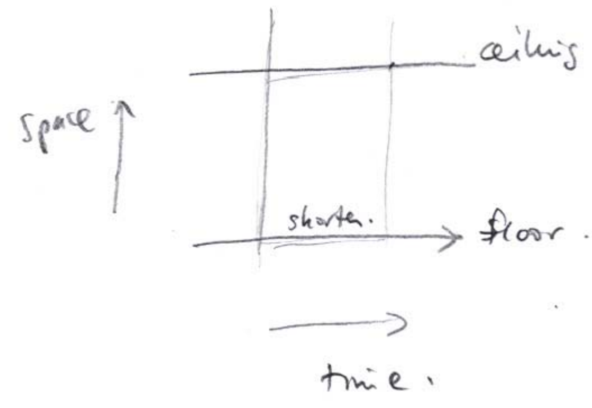
→ curvature.

(5)

Euclidean analogy.



space-$\underline{\underline{\text{time}}}$ curved.



pole $\Leftrightarrow$ black hole horizon.

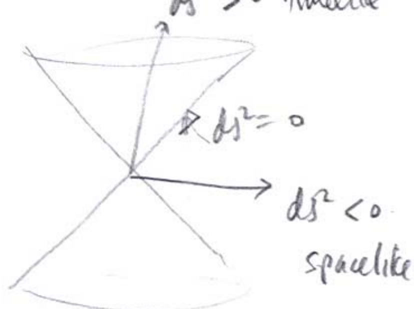
(return to paths later).

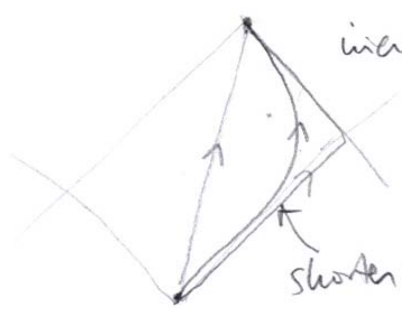
(6) $\underline{\text{metric:}}$ $\quad d\ell^2 \neq d\theta^2 + d\varphi^2$

$$d\ell^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

straight line: $\underline{\text{shortest}}$ path.

$\underline{\text{flat spacetime:}}$ $\quad ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$





inertial motion: $\underline{\text{longest}}$ path between nearby points

$\frac{(10\, m/s^2)}{10^{17} m/s^2} \sim \frac{10^{-16}}{m}$

curved: $\quad ds^2 = \underbrace{f(r)}_{1 - \frac{2GM}{rc^2}\,\Phi} dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$\rightarrow ds \sim \left(1 - \frac{GM}{rc^2}\right) dt \qquad \frac{d}{dr}\left(\frac{GM}{rc^2}\right) \cdot \Delta r = -\frac{gh}{c^2}$

(7) longest time: up and down.

turns out in einstein's theory predicts also:

$$\frac{1}{1-\frac{2GM}{rc^2}}\,dr^2 \;;$$

Light deflection depends on this, slow particle motion does not.

(8) general set-up. $ds^2 = \sum_{\alpha\beta=1}^{4} \boxed{g_{\alpha\beta}^{(x)}}\, dx^\alpha dx^\beta$

metric components.

Tricky b/c coordinate invariance.

eg. if $x^\alpha = x^\alpha(y)$, $dx^\alpha = \frac{\partial x^\alpha}{\partial y^\gamma} dy^\gamma$

$$ds^2 = \underbrace{g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^\gamma} \frac{\partial x^\beta}{\partial y^\delta}}_{g'_{\gamma\delta}} dy^\gamma dy^\delta$$

How much freedom? coordinates: 4 functions of 4 variables.

components: 10 functions of 4 variables.

(9) grav. field? $\cancel{\phi}$ $\cancel{\vec\nabla\phi}$ $\vec\nabla\vec\nabla\phi$ ✓

$\phi \updownarrow g_{\alpha\beta}$

$\vec\nabla\vec\nabla\phi \updownarrow$ " $\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial x^\delta} g_{\alpha\beta}$ "

⇝ curvature tensor

(10) field equation? $\nabla^2\phi = 4\pi G\rho_m$

$$\longleftrightarrow \quad G_{\alpha\beta} = \frac{8\pi G}{c^2} T_{\alpha\beta}$$

$(g, \partial g, \partial\partial g) \leftarrow$ Einstein tensor

energy-momentum tensor (current density)

$$R_{\alpha\beta} = \frac{8\pi G}{c^2}\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right)$$

$$= \frac{4\pi G}{c^2}\left(2T_{\alpha\beta} - T g_{\alpha\beta}\right)$$

like maxwell's displacement current

perfect fluid: $T_{\alpha\beta} \longleftrightarrow \begin{pmatrix} \rho_e & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$, $T \leftrightarrow \rho - 3p$

local frame adapted to fluid

$[\rho_e] = [p]$

<u>time-time component:</u> $2\rho - (\rho - 3p) = \underline{\rho + 3p}$