Lecture 2 Phys 798C Fall 2025 — Electrodynamics of SC

Saturday, August 30, 2025 10:56 AM

Simple treatments → more sophisticated → most sophisticated (Time)

→ Landau Equations, Meissner Effect, Complete Diamagnetism

① ② $\vec{A} = -\Lambda \vec{J}_s$

Screening of B-field

The London Equations F. H. London

1st London Eq. "derive".

Newton's 2nd Law of motion for charge carrier mass m, charge e

Acted upon by an $\vec{E}$-field. $m \xrightarrow{e} \vec{E}$, $\vec{F} = e\vec{E}$

Drude Model $\longrightarrow \dfrac{d(m\vec{v})}{dt} = e\vec{E} - \dfrac{m\vec{v}}{\tau}$ ← momentum relaxation time

$m \xrightarrow{\vec{v}_i} \oplus$, $\vec{v}_f$

Ignores Fermi-Dirac statistics

Single particle treatments,

Steady state equilibrium $\langle\ \rangle$

$$\left\langle \frac{d(m\vec{v})}{dt} \right\rangle \simeq 0 = e\vec{E} - \frac{m\langle \vec{v} \rangle}{\tau} \Rightarrow \boxed{\langle \vec{v} \rangle = \frac{e\vec{E}\tau}{m}}$$

$\vec{J} = ne\langle \vec{v} \rangle$ n = number density $= \dfrac{N}{V}$

$\vec{J} = \dfrac{ne^2\tau}{m}\vec{E} = \sigma_n \vec{E}$ Ohm's Law

Normal metal Apply $\vec{E} \Rightarrow$ Get $\vec{J}_n = \sigma_n \vec{E}$

→ $\tau \to \infty$ for a SC. Tinkham page 5

$$\frac{d}{dt}(m\vec{v}) = e\vec{E} \Rightarrow \boxed{\frac{d}{dt}\vec{J}_s = \frac{n_s e^2}{m}\vec{E}}$$

$\vec{J}_s = n_s e \langle \vec{v}_s \rangle$ supercurrent density

$\Lambda = \dfrac{m}{n_s e^2} = \mu_0 \lambda_L^2$ $\lambda_L = \sqrt{\dfrac{m}{\mu_0 n_s e^2}}$ London Penetration Depth

$$\boxed{\frac{d}{dt}(\Lambda \vec{J}_s) = \vec{E}}$$

Electric field accelerates the SC electrons

DC ⇒ infinite current.

AC current in a SC requires an AC electric field.

$$\frac{d}{dt}\vec{J}_s = \frac{1}{\Lambda}\vec{E} = \frac{1}{\mu_0 \lambda_L^2}\vec{E}$$

⇒ Screening of magnetic field.

Ampere's Law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_s + \cancel{\mu_0 \dfrac{\partial \vec{D}}{\partial t}} \to 0$ $f \lesssim 1\,\text{THz}$ $\left(\dfrac{2\Delta}{h}\right) \sim \text{THz}$

Displacement current

$\nabla \times \vec{B} \simeq \mu_0 \vec{J}_s$

Take the time derivative:

$$\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \frac{\partial \vec{J}_s}{\partial t} = \frac{1}{\lambda_L^2}\vec{E}$$

Now take the curl of both sides

$$\vec{\nabla} \times \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \frac{1}{\lambda_L^2}\underbrace{\vec{\nabla} \times \vec{E}}_{-\frac{\partial \vec{B}}{\partial t}}$$

$$\vec{\nabla} \times \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{1}{\lambda_L^2}\frac{\partial \vec{B}}{\partial t}$$

Integrate with respect to time

$$\underbrace{\vec{\nabla} \times \vec{\nabla} \times \vec{B}}_{\nabla(\cancel{\vec{\nabla} \cdot \vec{B}})^{\to 0} - \nabla^2 \vec{B}} = -\frac{1}{\lambda_L^2}\vec{B}$$

$$\boxed{\nabla^2 \vec{B} = \frac{1}{\lambda_L^2}\vec{B}} \to B \sim B_0 e^{\pm x/\lambda_L}$$

$\nabla^2 \vec{E} = \dfrac{1}{\lambda_L^2}\vec{E}$

$\lambda_L = \sqrt{\dfrac{m}{\mu_0 n_s e^2}}$ Aluminum $n = 18.1 \times 10^{22}\,\dfrac{1}{\text{cm}^3}$, $m = m_e$ } $\lambda_L = 12.5\,\text{nm}$

2nd London Equation

1st London Equation $\dfrac{\partial \vec{J}_s}{\partial t} = \dfrac{-n_s e^2}{m}\vec{E}$

Take the curl of both sides

$$\frac{\partial}{\partial t}\vec{\nabla} \times \vec{J}_s = \frac{-n_s e^2}{m}\underbrace{\vec{\nabla} \times \vec{E}}_{-\frac{\partial \vec{B}}{\partial t}}$$

$$\cancel{\frac{\partial}{\partial t}}\left[\vec{\nabla} \times \vec{J}_s + \frac{n_s e^2}{m}\vec{B}\right] = 0$$

Faraday's Law + Lenz's law oppose a change in flux

London Surmise (Let's pretend)

$$\vec{\nabla} \times \vec{J}_s + \frac{n_s e^2}{m}\vec{B} = 0$$

Meissner effect.

deGennes derivation in his book.

⇒ Consequence: To induce a current $\vec{J}_s$ in a Superconductor apply a magnetic field.

Limitations of the London Equations. Approximate

1) $B \ll B_c$ critical field Low-field limit.
2) Superfluid density uniform in space. Homogeneous SC
3) Local electrodynamics Drude. $\dfrac{d}{dt}(m\vec{v}) = e\vec{E} - \dfrac{m\vec{v}}{\tau}$

$\lambda_L \gg \dfrac{1}{\dfrac{1}{\xi_0} + \dfrac{1}{l_{MFP}}}$ $l_{MFP} = v_F \tau$

Normal Metal vs. Superconductor

SC	Normal Metal
Accelerates the current $\dfrac{\partial \vec{J}_s}{\partial t} = \dfrac{1}{\Lambda}\vec{E}$ persistent current with no electric field	$\vec{J}_n = \sigma_n \vec{E}$ E sustains a current
$\Lambda(\vec{\nabla} \times \vec{J}_s) = -\vec{B}$	$\vec{\nabla} \times \left(\dfrac{\vec{J}_n}{\sigma_n}\right) = -\dfrac{\partial \vec{B}}{\partial t}$ Faraday Ohm Lenz

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Simple London Equation $\vec{A}$ vector potential

$$\Lambda \vec{\nabla} \times \vec{J}_s = -\vec{B} = -\vec{\nabla} \times \vec{A}$$

$$\boxed{\vec{A} = -\Lambda \vec{J}_s}$$ + missing a term due to QM

macroscopic quantum phenomenon

$\partial/\partial t$

$\dfrac{\partial \vec{A}}{\partial t} = \vec{E} = -\Lambda \dfrac{\partial \vec{J}_s}{\partial t}$ 1st London Eq

$\vec{\nabla} \times$

$\vec{\nabla} \times \vec{A} = \vec{B} = -\vec{\nabla} \times (\Lambda \vec{J}_s)$ 2nd London Eq

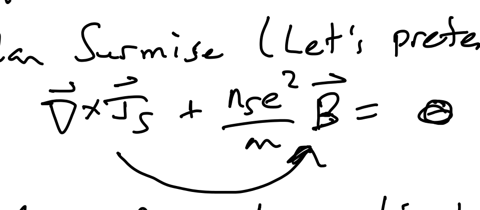
Gauge Choice London Gauge Choice

$\vec{A} \to \vec{A} + \vec{\nabla}\chi(\vec{r}) \Rightarrow$ same $\vec{B}$

London Gauge

1) $-\vec{A} = \Lambda \vec{J}_s$ Only works inside and on the surface of a SC
2) $\vec{\nabla} \cdot \vec{A} = 0 = \vec{\nabla} \cdot \vec{J}_s = 0$
3) $\vec{A} \to 0$ deep inside a SC (many λ_L)

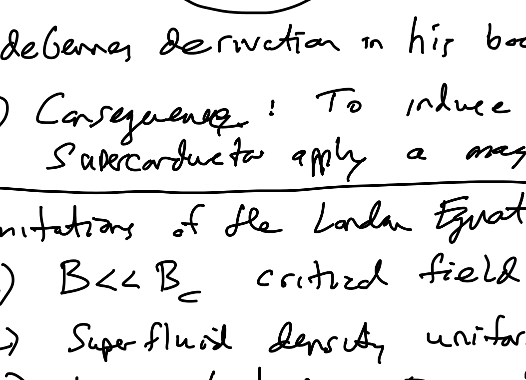
Boundary Condition of magnetic field on a SC



$H_\perp = 0$ on the surface of a SC.

Kittel son + Sang SC page 7.

Screening Properties of SCs.



xy-plane is a SC/vacuum interface

$B(z)$ inside the SC

$\nabla^2 \vec{B} = \dfrac{1}{\lambda_L^2}\vec{B}$ $\vec{B} = B(z)\hat{x}$

$\dfrac{d^2 B_x}{dz^2} = \dfrac{1}{\lambda_L^2}B_x$

$B_x(z) = \mu_0 H_a \left(c_1 e^{-z/\lambda_L} + \cancel{c_2}^{\,0} e^{+z/\lambda_L}\right)$ No divergence

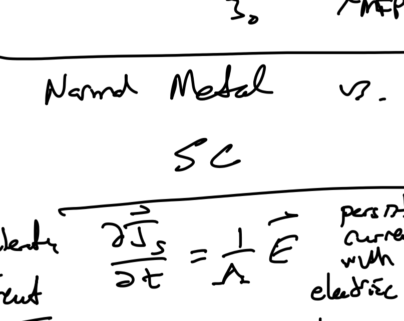
$B_x(z) = \mu_0 H_a e^{-z/\lambda_L}$ $B_x(0) = \mu_0 H_a$

$\mu_0 \vec{J}_s = \nabla \times \vec{B}$

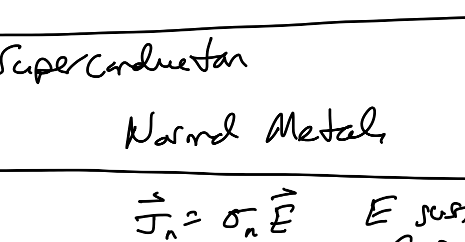
$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & 0 & 0 \end{vmatrix} = +\hat{j}\frac{\partial B_x}{\partial z}$$

$$\vec{J}_s = \frac{-1}{\cancel{\mu_0}}\frac{\cancel{\mu_0} H_a}{\lambda_L}e^{-z/\lambda_L}\hat{j} = -\frac{H_a}{\lambda_L}e^{-z/\lambda_L}\hat{j}$$

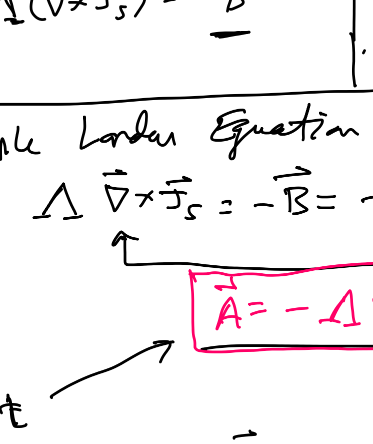
Diamagnetic screening current



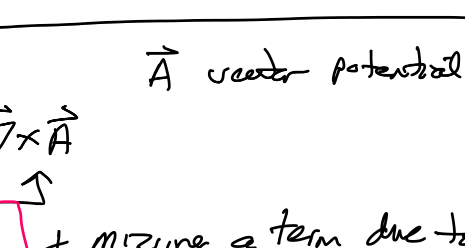
J_s on surface



Current-carrying wire DC current



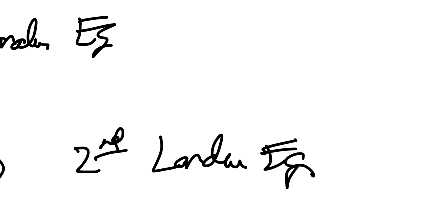
Normal Metal



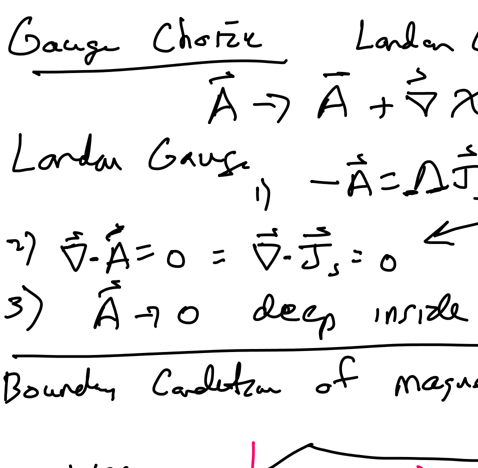
SC wire carrying J_s DC

$\nabla^2 J_s = \dfrac{1}{\lambda_L^2}\vec{J}_s$

$\nabla^2 \vec{B} = \dfrac{1}{\lambda_L^2}\vec{B}$



Meissner Effects



Strip conductor

Normal to SC contact

$\vec{\nabla} \cdot \vec{J}_s \neq 0$ Sink of supercurrent, source of supercurrent

Practical consequence