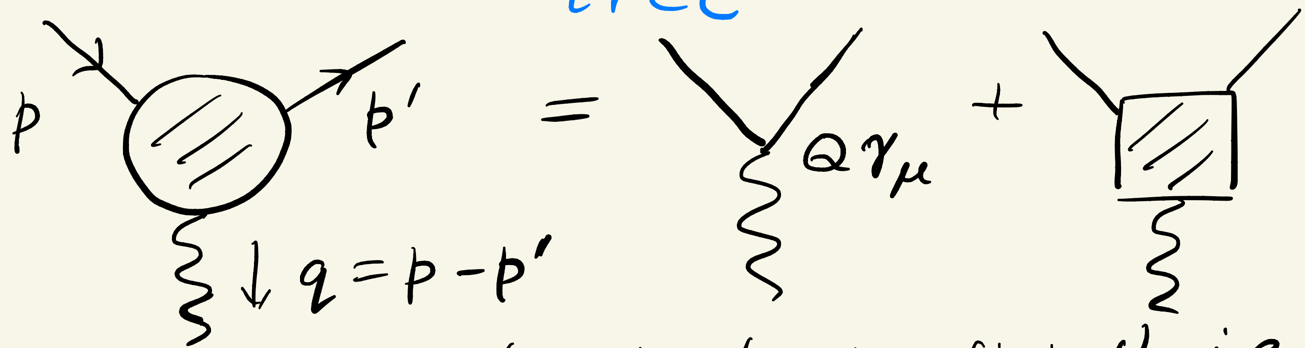
# Fermion-photon Vertex function:

$$\Gamma_\mu(p,p') = \underbrace{\gamma_\mu Q}_{\text{tree}} + \Gamma_\mu^{(\text{loop})}(p,p')$$




- Recall $D = 0$ (naive/superficial), i.e., log-divergence from general formula (all loops)
- Equivalently/more quickly, use dimensional analysis: $\left[\Gamma_\mu^{(\text{loop})}\right] = 0$, since can be thought of as coefficient of $\bar{\psi} A_\mu \psi$ in Lagrangian $\Rightarrow$ can't have $\Gamma_\mu^{(\text{loop})} \propto \Lambda^{>0} \times (\text{couplings})$ (assuming [couplings] $\geq 0$, i.e., renormalizable theory)

[e.g., can try $\Lambda^1/(m \text{ or } p \text{ or } q)$, but then $m, q, p \to 0$ limit doesn't make sense]

(dimensional "argument" valid for all interactions which are renormalizable contributing to $\boxed{\phantom{x}}$, e.g., Yukawa, i.e., $\phi$ in loop ... and all loop-order)

- In general (all loop), structure in Dirac space (like $\Sigma$): $\Gamma_\mu^{(\text{loop})} \propto \mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}$

(no $\gamma_5$ or $\gamma_\mu \gamma_5$ due to parity invariance)

- By Lorentz invariance, we must have (only momenta are $p_\mu$, $q_\mu$ or $p'_\mu$)

$$\Gamma_\mu^{(\text{loop})} \propto (p_\mu \text{ or } q_\mu)\mathbb{1} + \gamma_\mu + \sigma_{\mu\nu}(q^\nu \text{ or } p^\nu)$$

$\Rightarrow$ actual degree of divergence lowered by 1 (as compared to naive/superficial) for $\mathbb{1}$ & $\sigma_{\mu\nu}$ pieces, i.e., finite; only $\boxed{\gamma_\mu \text{ coefficient (log-) divergent}}$

- Above is for fermion(s)/photon being on or off-shell (general kinematics)...

...onto special case of $\boxed{\text{both fermions on-shell}}$ (photon must be off-shell, e.g., scattering of electron of off heavy nucleus by photon exchange), where, using Gordon identity & current conservation, we can re-write $\mathbb{1}$ term in terms of $\gamma_\mu$, $\sigma_{\mu\nu}$ giving

$$\boxed{\Gamma_\mu(p,p') = \gamma_\mu F_1(q^2) + i\sigma_{\mu\nu} q^\nu F_2(q^2)}$$

(see LP sec 11.1 or PS sec. 6.2)

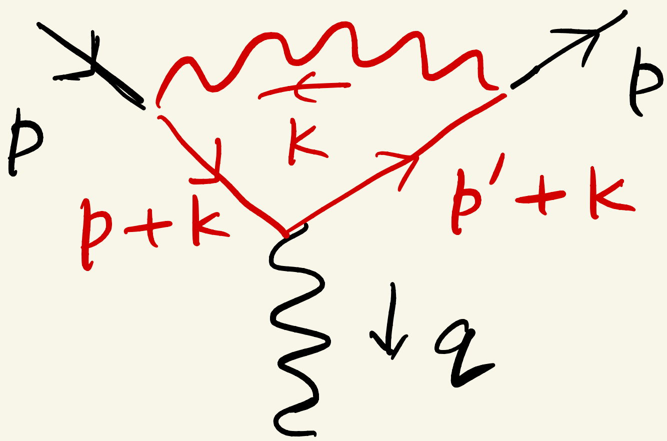
$$\Rightarrow F_1(q^2) = Q\,(\text{tree}) + \underbrace{F_1^{(loop)}(q^2)}_{(\log\text{-})\text{divergent}}$$

$\boxed{F_{1,2} \text{ are form-factors}}$

$F_2(0)$ [finite] is anomalous magnetic moment of fermion (sec. 11.2.2 of LP or sec. 6.2 of PS)

— Onto 1-loop: estimate gives log-divergence from $\int d^4k \underbrace{\frac{1}{k^2}}_{\text{photon}} \underbrace{\frac{1}{k}\,\frac{1}{k}}_{\text{fermion}}$

propagators

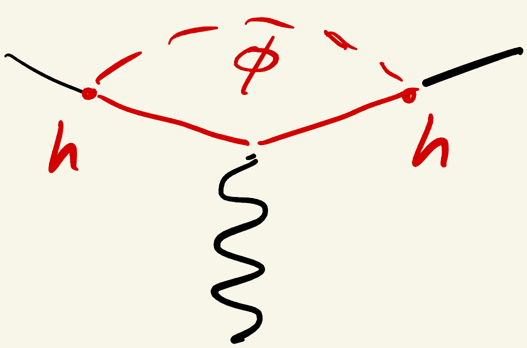



— Calculate (using DIMREG): why?!

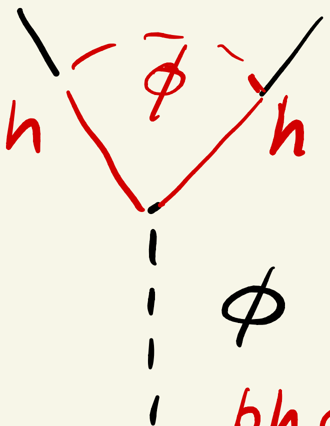
(i) practice; (ii) relation to $\Sigma$, based on gauge invariance, hence crucial to use DIMREG (cf. hard cut-off (see HW 2.1 : general setting-up in LP sec 11.3)

(iii) confirm no reduction of divergence (no "surprises") : after all, no principle at play here

– again, above arguments valid for photon-fermion vertex even with Yukawa (+ QED) interaction :




– And, since gauge invariance not used above (for $\Gamma_\mu$ by itself), we can generalize to vertex in pure Yukawa theory as well :




(cf. photon in QED)

$$\Gamma_{(\text{no }\mu)} \sim \underset{\substack{\uparrow \\ \text{log-divergent}}}{\mathbb{1}} + \underbrace{\underset{\times p^\mu \text{ or } q^\mu}{\gamma_\mu} + \underset{\times p^\mu q^\nu}{\sigma_{\mu\nu}}}_{\text{finite}}$$

— Recall: for $\boxed{\Pi_{\mu\nu}}$, gauge invariance (or WT identity) informs us beforehand that (all loops) $D = 2$ (superficial) $\rightarrow D = 0$ (actual), but estimate for 1-loop diagram still shows $D = 2$ : it's only after calculating, regularizing loop integral (that too, with DIMREG, not hard cut-off) that we explicitly see quadratic divergence disappear (again, lot of work!)...

while for $\boxed{\Sigma}$, chiral symmetry / Dirac structure tells us $D = 1$ (superficial) $\rightarrow D = 0$ (different principle than for $\Pi_{\mu\nu}$); then even (rough) estimate for 1-loop diagrams shows same, backed-up (if needed) by calculating loop integral, but without needing to regularize etc. (again, hard cut-off will also lower divergence, since gauge invariance is not really at play here)

Next step is adding $\boxed{\text{counterterm}}$

- following is sufficient, since divergence is only in $\gamma_\mu$ term (at all loops):

$$\mathcal{L}_{CT}^{(\Gamma_\mu)} = -Q(Z_1 - 1) e \bar{\Psi} \not{A} \Psi$$, again same form as $\mathcal{L}_{\text{classical}} = -eQ\bar{\Psi}\not{A}\Psi$

- Choice of kinematics for fixing $Z_1$:

both external fermions on-shell ($p^2 = p'^2 = m^2$) and $q^\mu$ (photon momentum, off-shell) $= 0$

$\Rightarrow$ initial & final fermion momentum/state is same!

So, really it's $q \to 0$ (specifically $\ll m$), e.g., non-relativistic / atomic system

In this situation, choose CT so that it **exactly** cancels loop contribution to $\Gamma_\mu$:

$$\Gamma_\mu^{\text{classical}} + \Gamma_\mu^{(\text{loop})}\begin{pmatrix} p^2 = p'^2 = m^2, \\ q \to 0 \end{pmatrix} + CT = \Gamma_\mu^{\text{classical}}$$

$$\left( = \gamma_\mu Q^{\text{classical}} \right)$$

Since for both fermions being on shell (but $q^2$ need not vanish) we have

$$\Gamma_\mu^{(\text{loop})}\left(p^2 = p'^2 = m^2\right) = \gamma_\mu F_1^{(\text{loop})}(q^2) + i q^\nu \sigma_{\mu\nu} F_2(q^2)$$

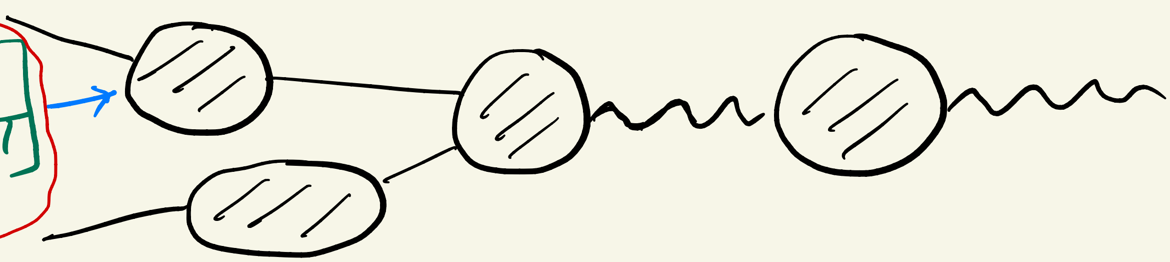
we get $\boxed{F_2^{(\text{loop})}(q^2 = 0) = -Q(Z_1 - 1)}$

Comments on / interpretations of choice of $Z_1$

– considering non-relativistic form of spinors, we can show (see LP sec. 11.2.1 or PS sec. 6.2) that in above limit ($p^2 = p'^2 = m^2$; $q \to 0$), only coupling to electrostatic potential ($\phi$) surives (again, from $\bar{u}\, \Gamma_\mu u$), i.e., coupling to magnetic vector potential ($\bar{A}$) requires at least one power of $q$ ...

$\Rightarrow$ above (kinematic) configuration measures electric charge, so that's what then seems to be chosen to be same as classical ... but just to complete the picture, note that full photon propagator in this limit ($q^2 \to 0$) was also "chosen" (via $Z_3$, i.e., CT for $\Pi_{\mu\nu}$) to be same as classical and dressing (self-energy) of external fermion vanishes (choice of $\Sigma_{CT}$): schematically

[tree + loop + CT] = tree



$\Rightarrow$ "full" coupling to EM field is same as classical in this case, i.e., $\boxed{eQ}$, so that "e" here is as measured in (e.g.) atomic systems ($q^2 \to 0$ limit), i.e., $e^2/(4\pi) = \alpha_{QED} = \frac{1}{137}$

[again, effective coupling for $q^2 \neq 0$, e.g., higher energies, will then be different: we will calculate this running later ...]

– Expanding around $q_\nu = 0$, i.e., keeping leading terms for $q \neq 0$, we obtain coupling to magnetic field: $F_1(0)$ gives Dirac magnetic moment, $g = 2$, i.e., same as tree, since

$$F_1^{full}(0) = Q\ (\text{tree}) + F_1^{loop}(0) + Q\,(Z_1 - 1)\ \text{from CT}$$
$$= Q \quad (\text{CT chosen to cancel loop})$$

while $F_2(0)$ is anomalous magnetic moment [or $(g-2)$]. [$F_{1,2}\,(q^2 \neq 0)$ will give even higher-order effects]

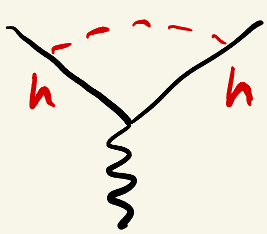
– $\boxed{\text{WT identity} \Rightarrow Z_1 = Z_2}$ :

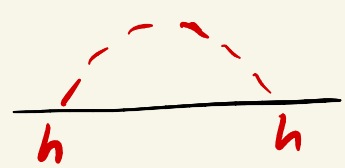
(see separate note for proof)

Comments on $Z_1 = Z_2$ : since it's based on QED gauge invariance, we see that

(1). it will not be valid if we use regulator (such as hard cut-off) which breaks gauge invariance : see HW 2.1

(2). it holds (for photon vertex) even if we use Yukawa interaction (which is QED gauge-invariant) inside loop, i.e.,

$Z_1$ "from" 

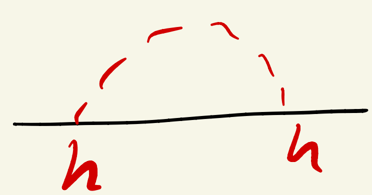
vs. $Z_2$ from 


(Note, $h$ is different for electron vs. muon ...)

(3). it's valid for all choices of gauge fixing parameter ($\xi$), since we didn't have to specify that in proving WT identity

However, $Z_1$ (or $Z_2$) - by itself - is dependent on $\xi$ : in fact, for $\xi = 0$ (Landau gauge), it is finite ...again, such that $Z_1 = Z_2$ always...

(4). an analogous relation does not hold for pure Yukawa theory (since we do<u>n</u>'t have WT identity in that case), i.e.,

"$Z_1$" from  $\neq$ $Z_2$ from 



(not photon)



(5). We will show later (next topic) that $Z_1 = Z_2$ ensures ratio of electric charge of electron to muon or proton is not modified (vs. classical value) by loops, even if interactions in loops (e.g., Yukawa coupling) are different for electron vs. muon/proton ...

... whereas – given #4 above – ratio of Yukawa coupling of electron to muon ... is modified at loop-level (we'll come back to this point at end of course in the context of GUT's !)

– Finally, check that $\left(\Gamma_\mu^{(\text{loop})} + CT\right)$ is finite (divergences in $\Gamma_\mu^{(\text{loop})}$ cancelled by CT) even if fermions off-shell and $q \neq 0$ (general case): a bit subtler than for $\Pi_{\mu\nu}$ & $\Sigma$, since there are more terms

( Dirac structure) in this situation:

$$\Gamma_\mu(\text{general}) = \gamma_\mu f_1 + \mathbb{1}\, p_\mu f_2 + \mathbb{1}\, f_3\, p'_\mu + \sigma_{\mu\nu}\, p^\nu f_4 + \sigma_{\mu\nu}\, p'^\nu f_5$$

[cf. $\Gamma_\mu = \gamma_\mu F_1(q^2) + i\sigma_{\mu\nu} q^\nu F_2(q^2)$

for both fermions being on-shell, but not photon $(q \neq 0)$

$\longrightarrow \gamma_\mu F_1(0)$ for $q \to 0$ (e.g., atomic systems)], as assumed while choosing CT

– See separate note: essentially, divergence only in $F_1(0)$...