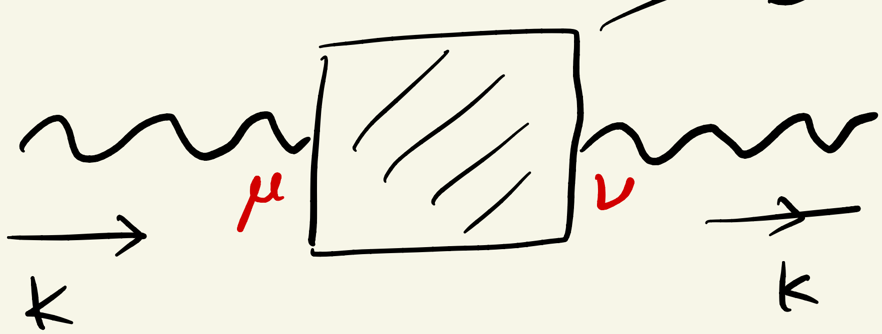
Vacuum polarization (self-energy of photon): main results are

(i). photon remains massless at loop-level

(ii). Running of QED coupling originates in $\Pi_{\mu\nu}$, but will have to deal with $\Gamma_\mu^{(loop)}$ & $\Sigma$ before getting there...

- $\Pi_{\mu\nu}(k)$ is 1PI loop correction to photon 2-point function: $k$ is "external" photon momentum, but can be off-shell: $k^2 \neq 0$ (don't count polarization vector, $\varepsilon_{\mu,\nu}$ of photons)

1PI



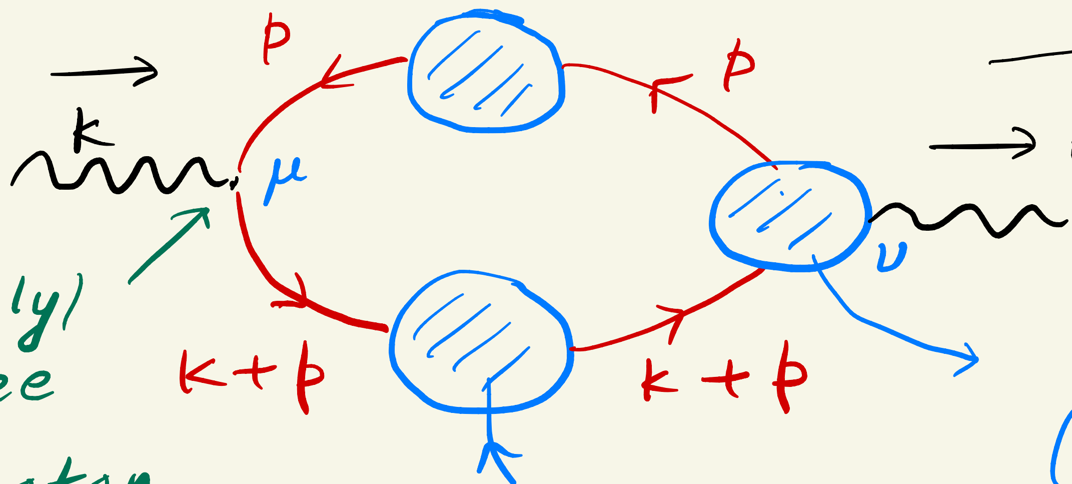

- $\boxed{\text{First}}$, use WT identity to get form of $\Pi_{\mu\nu} = \pi(k^2)(g_{\mu\nu}k^2 - k_\mu k_\nu)$:

$D = 2$ (naively quadratically divergent)

$\longrightarrow D = 0$ (logarithmic), due to "symmetry" requiring 2 powers of external momentum

– In general, $\Pi_{\mu\nu}(k)$ looks like (see diagrams at end of note)

(p is loop momentum)




(only) tree (photon "has" to start this way)

tree+loop (full vertex)

tree + loop (full fermion propagator)

$\Longrightarrow$

$$i\,\Pi_{\mu\nu}(k) = \int \frac{d^4p}{(2\pi)^4} \times (-1)\ (\text{fermion loop}) \times$$

$$\text{Trace}\Big[\underbrace{ieQ\gamma_\mu}_{\substack{\text{tree}\\ \text{on LHS}}}\ \underbrace{iS_F(p)}_{\text{full}}\ \underbrace{ie\Gamma_\nu(k+p,p)}_{\substack{\text{full}\\ \text{on RHS}}}\ \underbrace{iS_F(k+p)}_{\text{full}}\Big]$$

Multiply by $k_\nu$ (contracts with $\Gamma_\nu$):

use $\boxed{WT}$ identity in $1^{st}$ form:

note

$$k^\nu \Gamma_\nu(k+p,p) = Q\left[S_F^{-1}(k+p) - S_F^{-1}(p)\right]$$

photon momentum

"cancels" 1 of 2 $S_F$'s in $\Pi_{\mu\nu}$

$$k^\nu \Pi_{\mu\nu}(k) = i(eQ)^2 \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\left[\gamma_\mu \left\{S_F(p) - S_F(k+p)\right\}\right]$$

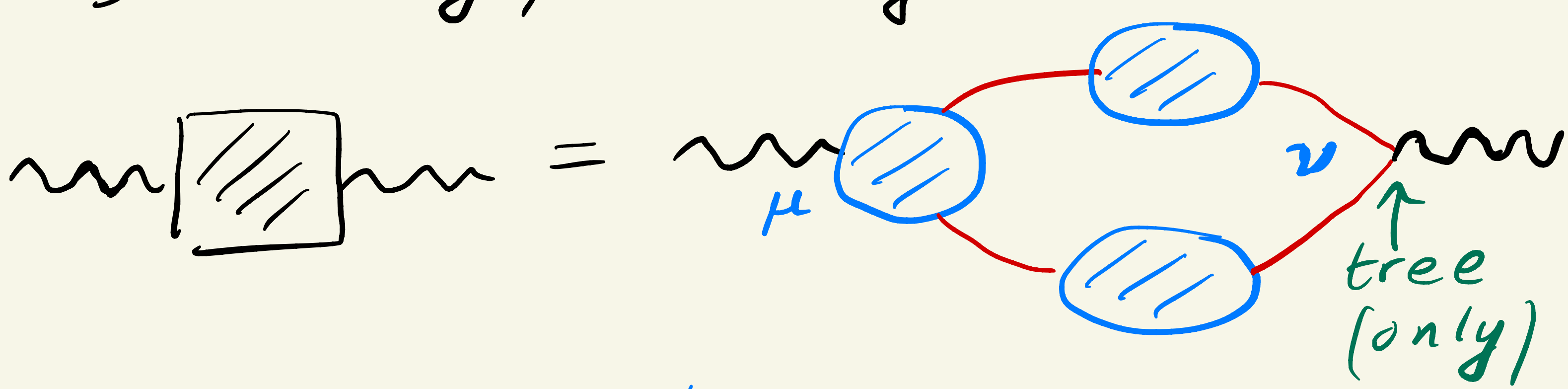
However, $p$ is "dummy" variable (integrated over),

so $(k+p) \longrightarrow p$ in $2^{nd}$ term gives

$$\boxed{k^\nu \Pi_{\mu\nu}(k) = 0}$$

(again, due to WT identity/ gauge invariance)

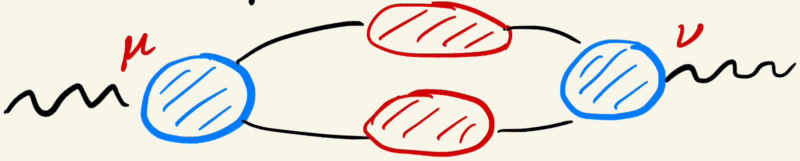
– Similarly, thinking of (same)




(tree-only at $\nu$/RHS vs. $\mu$/LHS earlier) gives $k^{\mu} \Pi_{\mu\nu} = 0$

(can actually show for amplitude with $>2$ external photons also: see PS sec. 7.4)

– Note: what about at both external photon vertices ($\mu$ & $\nu$)?!




... or no at these vertices...

..."double counting", so no need to "add"

$$\Rightarrow \underbrace{\Pi_{\mu\nu}(k)}_{\text{all loops}} \overset{\text{(must be)}}{“=”} \underbrace{\Pi(k^2)}_{\text{Lorentz scalar}} \boxed{\left(g_{\mu\nu} k^2 - k_\mu k_\nu\right)}$$

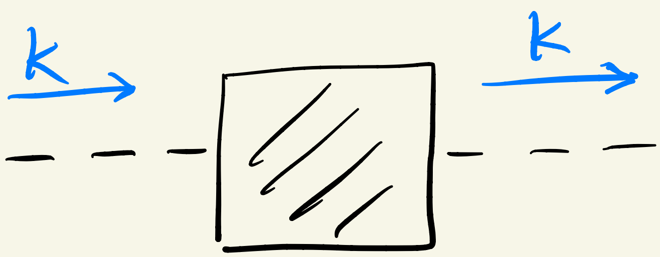
only tensor giving 0 upon contracting with $k_\mu$ & $k_\nu$ (no other momentum)

So, 2 external photons (no external fermions) $\Rightarrow$ amplitude $\propto$ 2 powers of external momenta $\Rightarrow$ 2 powers less of loop momenta ...

$D = 2$ (superficial) $\longrightarrow$ $D = 0$ (actual)

... contrast with scalar ($\phi$) case, say, in Yukawa theory:

$\mathcal{L}_{int} \sim h \, \phi \bar{\Psi} \Psi$




$\cancel{\propto} \; k^2$ will remain quadratically divergent ($D = 2$ not reduced, since no analog of WT identity / gauge invariance)

— $\boxed{\text{Next}}$, use above form of $\Pi_{\mu\nu}$ to obtain full photon propagator, showing photon remains massless at loop-level ...

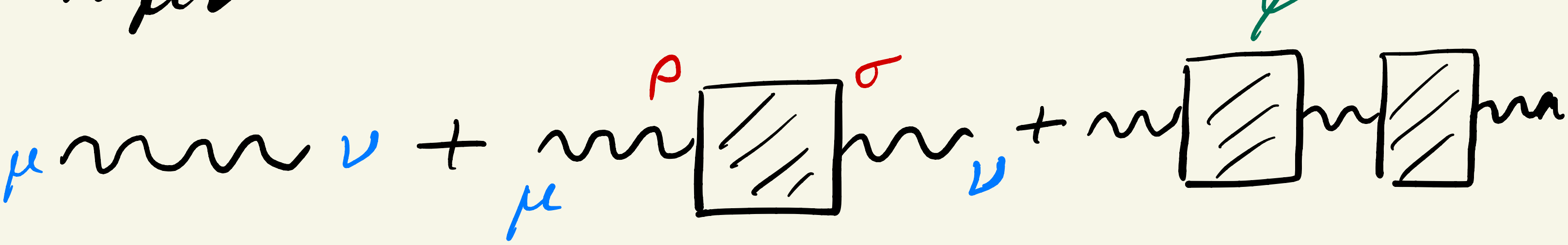
... recall: zero mass for photon at tree-level "related to" gauge invariance

$\Rightarrow$ (assuming gauge invariance intact at loop-level), photon should remain massless at loop-level ...

... but good to check expectation explicitly as follows.

– Like for fermion propagator, full photon propagator by re-summing $\Pi_{\mu\nu}$ insertions :

1 PI loop (only)




(use 't Hooft-Feynman gauge for tree-level propagator)

– schematically, (dropping Lorentz ...) tensor

$$\frac{1}{k^2} + \frac{1}{k^2}\,\Pi_T\,\frac{1}{k^2} + \frac{1}{k^2}\,\Pi_T\,\frac{1}{k^2}\,\Pi_T\,\frac{1}{k^2}$$

$$\sim \text{(geometric series)} \quad \frac{1}{\left[k^2 - \Pi_T(k^2)\right]}$$

$$\sim \frac{1}{k^2\left[1-\pi_S(k^2)\right]}$$ , since $\boxed{\pi_T \sim k^2 \pi_S}$

scalar

again, using above result (from WT identity)

$\Rightarrow$ massless pole in propagator survives due to $\pi_{\mu\nu} \propto (k^2 g_{\mu\nu} - k_\mu k_\nu)$, in turn WT identity/gauge invariance

[In detail, see PS Eq. 7.74:

$$\text{sum} = \left(\frac{-ig_{\mu\nu}}{k^2}\right) + \left(\frac{-ig_{\mu\rho}}{k^2}\right)\underbrace{\left(i\pi^{\rho\sigma}\right)}_{\substack{(k^2 g^{\rho\sigma} - k^\rho k^\sigma)\,\pi(k^2) \\ \text{(all loops)}}}\left(\frac{-ig_{\sigma\nu}}{k^2}\right) + \ldots$$

$$= \frac{-ig_{\mu\nu}}{k^2} + \frac{-ig_{\mu\rho}}{k^2}\underbrace{\Delta^\rho{}_\nu}_{\delta^\rho{}_\nu - k^\rho k_\nu / k^2}\pi(k^2) - \frac{ig_{\mu\rho}}{k^2}\Delta^\rho{}_\sigma \Delta^\sigma{}_\nu \left[\pi(k^2)\right]^2 + \ldots$$

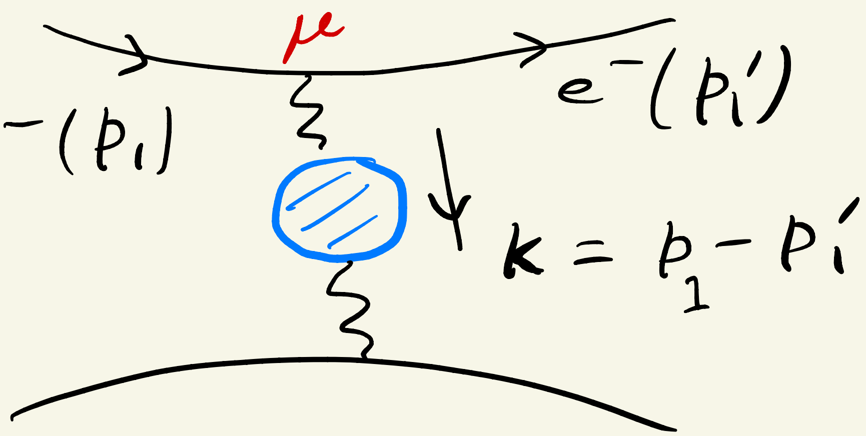
$$= -\frac{i g_{\mu\nu}}{k^2} - \frac{i g_{\mu\rho}}{k^2}\left(\delta^\rho_\nu - \frac{k^\rho k_\nu}{k^2}\right)\left\{\pi(k^2) + \left[\pi(k^2)\right]^2 + \cdots\right\}$$

$$= \frac{-i}{k^2\left[1-\pi(k^2)\right]}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{-i}{k^2}\frac{k_\mu k_\nu}{k^2}$$

massless pole

"Extra" pieces $\propto k_\mu k_\nu$ don't contribute to S-matrix element, since "dot" into fermion "current" (see PS Eq. 7.74 or LP Eq. 9.32 or PS below Eq. 9.58 for general argument),

e.g.




so that $k^\mu$ from re-summed photon propagator multiplies $\bar{u}(p_1')\gamma_\mu u(p_1)$

giving $\bar{u}_1(p_1')\left(\underbrace{\overset{m}{\not p_1}}_{} - \not p_1'\right) u(p_1) = 0$

$m$ $m$

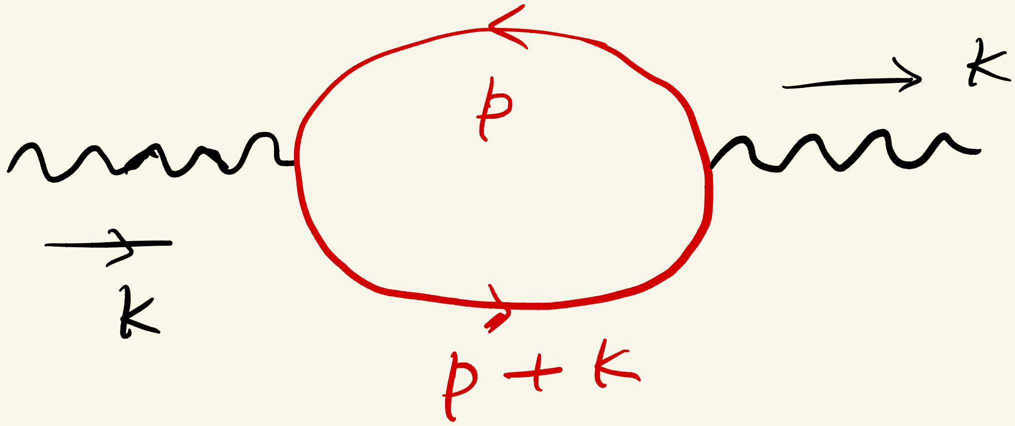
... so $k_\mu k_\nu$ terms in full photon propagator can be "dropped" ... ]

— Again, above argument does not apply to scalar propagator (no gauge invariance to "protect") ⇒ even if scalar mass $= 0$ at tree-level, it will acquire non-zero mass at loop-level, i.e., in analogy with photon/fermion propagator, we get full/resummed scalar (massless at tree-level) propagator $\sim \frac{1}{k^2 - \Sigma_{scalar}}$, but unlike photon case, $\Sigma_{scalar} \not\propto k^2$, so no massless pole at loop-level.

— Onto, one-loop calculation ...

# $\Pi_{\mu\nu}(k^2)$ at one-loop in QED

— (Naive) estimate first:




$$\sim \int d^4p \, \frac{1}{(\not{p} - m)} \, \frac{1}{(\not{p} + \not{k} - m)}$$

Since we are looking for UV divergences, we can drop $m$ and expand $2^{nd}$ propagator for $p \gg k$:

$$\frac{1}{(p+k)} \sim \frac{1}{p} + \frac{k}{p^2} + \frac{k^2}{p^2} + \dots$$

giving $\sim \int \frac{d^4p}{p^2} + \int d^4p \, \frac{k}{p^3} + k^2 \int \frac{d^4p}{p^4}$

where $1^{st}$ term is naive quadratic divergence ($D = 2$), which should vanish based on above form of $\Pi_{\mu\nu}$, but not "obvious" when we calculate!

$2^{nd}$ term: integrates to zero due to odd (net) powers of $p$ ...

$3^{rd}$ term gives log-divergence ($D = 0$) and expected dependence on external momentum

– explicit calculation using DIMREG (done in LP, but here will delineate steps): why do it when we know form of result (photon remains massless at loop-level)?!

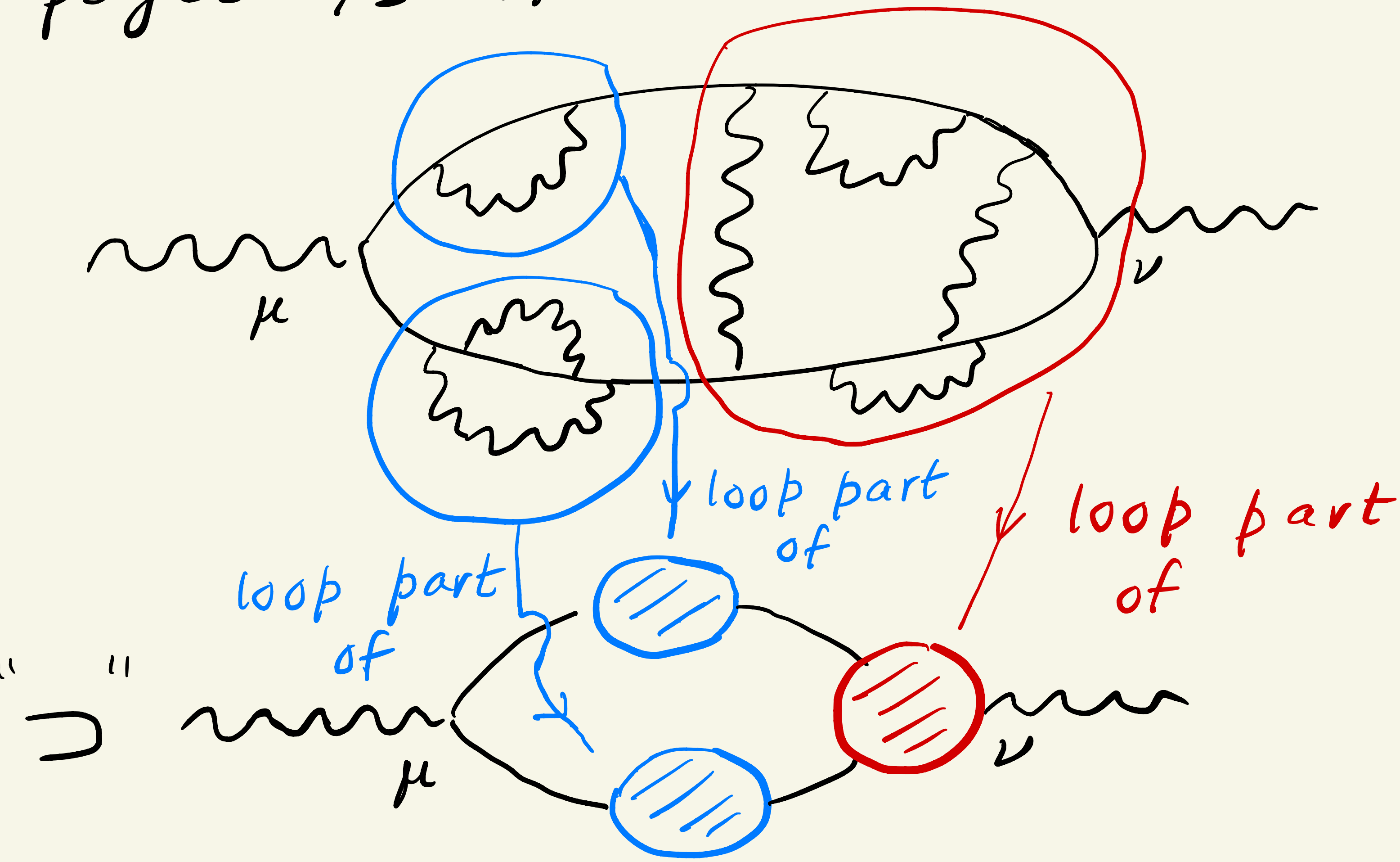
Motivation: (i) illustrate need to use "proper" regulator (DIMREG respects gauge invariance, whereas hard momentum cut-off does not)

... not so obvious until we actually calculate (even with DIMREG vanishing of $D = 2$ is subtle)

(ii) running of QED coupling will arise from $\Pi_{\mu\nu}$

(iii) practice: tricks/manipulations used here valid for other loop diagrams also

Example of how a generic loop diagram can be "recast" into form shown on pages 2,3 of this note




or (same diagram)