$\Pi_{\mu\nu}(k)$ using DIMREG

(1st step / regularization)

- Look out for how $(k^2 g_{\mu\nu} - k_\mu k_\nu)$ structure emerges : non-trivial, even if was expected based on WT identity / gauge invariance

- DIMREG : number of spacetime dimensions $= (4 - 2\,\varepsilon) \equiv N \quad (\varepsilon > 0)$
as per LP (PS has "1")

$\Rightarrow [\mathcal{L}] = 4 - 2\varepsilon \Rightarrow [\psi] = \frac{3}{2} - \varepsilon \,;$

$[A_\mu] = 1 - \varepsilon$ (from kinetic terms) $\Rightarrow$

$\mathcal{L}_{int} \sim e\,\mu^{\varepsilon}\,\bar{\psi}\,\not{A}\,\psi$, where

$[e\,(\text{usual})] = 0$ (since $[\bar{\psi}\not{A}\psi] = 4 - 3\varepsilon$)

- Roughly / intuitively, net coupling constant mass dimension > 0, even

if "epsilonically" vs. $=0$ in 4d...
UV divergences "less" severe in general, since $D \ni \cdots \boxed{-\delta_i} \cdots$

$\Rightarrow$ in addition to unphysical parameter ε, introduce μ (arbitrary mass scale / subtraction point) : check later that physical amplitudes are independent of ε & μ

step "0": $$i\,\Pi_{\mu\nu}(k) = \int \frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}} \,(-1)$$

$$\times \, \mathrm{Tr}\left[i e \mu^{\varepsilon} \gamma_\mu \frac{i(\not{p}+m)}{p^2-m^2} \, i e \mu^{\varepsilon} \gamma_\nu \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2} \right]$$

step $\boxed{1}$: keep even powers of γ's in numerator product, since $\mathrm{Tr}[\gamma^{odd}]=0$

$$\Pi_{\mu\nu}(k) = i e^2 \mu^{2\varepsilon} \int d^{4-2\varepsilon}p \,/\, (2\pi)^{4-2\varepsilon}$$

$$\frac{\mathrm{Tr}\left[\gamma_\mu \not{p} \gamma_\nu (\not{p}+\not{k}) + m^2 \gamma_\mu \gamma_\nu\right]}{\left[(p+k)^2 - m^2\right]\left[p^2 - m^2\right]}$$

Step ②: Use Feynman parameters to combine denominators (LP sec. 11.3 ; PS sec. 6.3 : mathematical "trick"):

$$\frac{1}{\left[(p+k)^2 - m^2\right](p^2 - m^2)} = \int_0^1 dx \frac{1}{\left\{(1-x)(p^2-m^2) + x\left[(p+k)^2 - m^2\right]\right\}^2}$$

$$= \int_0^1 dx \frac{1}{\left[p^2 + 2p\cdot k\, x - m^2 + k^2 x\right]^2}$$

complete square
$$= \int dx \frac{1}{\left[(p+kx)^2 - m^2 + k^2 x(1-x)\right]^2}$$

Step $\boxed{3}$: shift (loop) momentum :

$p' = p + kx$ (also in trace of numerator)

$$\Pi_{\mu\nu} \propto \int d^{4-2\varepsilon}p' \int_0^1 dx \frac{1}{(p'^2 - a^2)^2}$$

$$\times \mathrm{Tr}\left\{\gamma_\mu (\not{p}' - \not{k}\,x)\gamma_\nu [\not{p}' + \not{k}(1-x)] + m^2 \gamma_\mu \gamma_\nu\right\}$$

where $\boxed{a^2 \equiv m^2 - k^2 x(1-x)}$

... then $p' \to p$ (dummy variable)

Step 4 : drop odd powers of p in numerator, since denominator is even powers, so integrates to zero (more clear after Wick rotation: see step 8)

$\Rightarrow$ relevant numerator,

$$N_{\mu\nu} = \mathrm{Tr}[\gamma_\mu \not{p}\, \gamma_\nu \not{p}] + m^2\,\mathrm{Tr}[\gamma_\mu \gamma_\nu]$$
$$- x(1-x)\,\mathrm{Tr}[\gamma_\mu \not{k}\, \gamma_\nu \not{k}]$$

Step $\boxed{5}$: While the number of Dirac γ- matrices can be analytically continued to non-integer, i.e., γ_μ, with $\boxed{\mu = 0 \text{ to } 3-2\varepsilon}$ (cf. $\gamma_{0,1,2,3}$ in 4d), it seems that dimension of γ- matrices can not be continued in this manner. So, we keep γ- matries of dimension 4, i.e., $\mathrm{Tr}\, \mathbb{1} = 4$;

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \mathbb{1}_{4\times 4};$$

$$\gamma^\mu \gamma_\mu = \underbrace{(4-2\varepsilon)}_{g^{\mu\nu} g_{\mu\nu}} \mathbb{1}_{4\times 4} \ldots$$

So, as in 4d, $\mathrm{Tr}\left[\gamma_\mu \gamma_\nu\right] = 4 g_{\mu\nu}$;

$$\mathrm{Tr}\left[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho\right] = 4\left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho} + g_{\mu\rho} g_{\nu\lambda}\right)$$

Thus, after some algebra,

$$N_{\mu\nu} \text{ (numerator)} = -8x(1-x)\left(k_\mu k_\nu - k^2 g_{\mu\nu}\right) + 4\left[2\boxed{p_\mu p_\nu} + g_{\mu\nu}\left(a^2 - p^2\right)\right]$$

Step 6 : Simplification upon $\int d^{4-2\varepsilon}p$

In general, $I^{\nu\rho} = \int d^{4-2\varepsilon}p \; \underbrace{p^\nu p^\rho f(p^2)}_{\text{symmetric in } \nu,\rho}$

$$(\text{must be}) = g^{\nu\rho} J \leftarrow \text{constant}$$

Calculate "J" : $g_{\nu\rho} \times \ldots$ on both sides:

$$\int d^{4-2\varepsilon}p \; p^2 f(p^2) = \underbrace{g^{\nu\rho} g_{\nu\rho}}_{N = 4-2\varepsilon} J$$

$$\Rightarrow \quad I^{\nu\rho} = \frac{g^{\nu\rho}}{N} \int d^{4-2\varepsilon}p \; p^2 f(p^2)$$

Use it on 2^{nd} term in $N_{\mu\nu}$ above:

after $\int d^{4-2\varepsilon}p$

$$N_{\mu\nu} = 8x(1-x)\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right)$$

} form expected (using WT identity)

$$+ 4 g_{\mu\nu}\left[\left(\frac{2}{N} - 1\right)p^2 + a^2\right]$$

$\rightarrow$ "unwanted" form

$\Rightarrow$

2nd term / naively quadratically divergent/ should $\boxed{\text{integrate}}$ to 0 (this is at 1-loop, but valid for all orders as per theorem)...

step $\boxed{7}$ Using DIMREG (not obvious, but) intergal of 2nd term (after Wick rotation: see next step) does vanish: see LP Eq. 12.47 for details or PS bottom of page 251 (mostly "math")...

... cf. hard cut-off regularization:

$p^2_{E(\text{uclidean})} \leq \Lambda^2_{UV}$, analog of

→ see below

2nd term will not vanish, i.e., will not get purely $(k^2 g_{\mu\nu} - k_\mu k_\nu)$ structure

– physics "interpretation": DIMREG preserves gauge invariance / WT identity, while hard cut-off does not (see also HW 2.1 for calculation of Γ_μ vs. Σ)

– "Intuition": analytic continuation

of number of space-time dimensions should not violate gauge invariance:

$$\psi(x) \longrightarrow e^{-ieQ\,\theta(x)}\psi(x) \;\&\; A_\mu \rightarrow A_\mu + \partial_\mu \theta$$

However, hard cut off has the following "problem". Note that while $p_E^2 \leq \Lambda_{UV}^2$ restricts "frequencies" (Fourier modes) contained in the fields $A_\mu(x)$, $\psi(x)$, the gauge transformation parameter (not a field), $\theta(x)$ is still arbitrary, i.e., can contain frequencies above Λ_{UV}. So, such gauge transformation could lead to new fields having frequencies violating cut-off, resulting in a incomptability.

Step 8: Wick rotation (mainly "math trick": see Eq. 11.61 of LP or Eq. 6.48 of PS) so that $\int d^4p$ "simplifies" to angular & radial integration (4d "spherical" coordinates for momentum)

[Need to be careful with poles: see references...]

Recall, we have so far

$$\Pi_{\mu\nu}(k) = 8\, i\, e^2 \mu^{2\varepsilon} \left(k^2 g_{\mu\nu} - k_\mu k_\nu\right) \times$$

$$\int \frac{d^{4-2\varepsilon} p}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx \, \frac{x(1-x)}{\left[p^2 - a^2\right]^2},$$

where p^2 (Minkowskian) $= (p^0)^2 - |\bar{p}|^2$.

Go to $\boxed{\text{Euclidean}}$ momentum: $p^0 = i\, p^0_E$

so that $p^2 = - p_E^2 = -\left[(p^0_E)^2 + |\bar{p}|^2\right]$,

$\bar{p}$ unchanged

$\Rightarrow (p^2 - a^2)$ in integrand $= -(p_E^2 + a^2)$

Using sec. A.4 of LP for integration, we get (after some algebra)

$$\Pi_{\mu\nu}(k) = \frac{-8 e^2}{16\pi^2} \left(g_{\mu\nu} k^2 - k_\mu k_\nu\right) \Gamma(\varepsilon)$$

function (not vertex!)

$$\int_0^1 dx \left(\frac{\mu^2 4\pi}{a^2}\right)^{\varepsilon} x(1-x)$$

$a^2 \longrightarrow m^2 - x(1-x)k^2$

Step $\boxed{9}$: Expand for small ε :

$$\Gamma(\varepsilon) = \underbrace{\frac{1}{\varepsilon}}_{\substack{\text{divergence} \\ \text{as } \varepsilon \to 0 \text{ (physical limit)}}} - \underbrace{\gamma_E}_{\text{Euler-Mascheroni constant} \simeq 0.58} + \mathcal{O}(\varepsilon)$$

and $\left(\frac{\mu^2 4\pi}{a^2}\right)^{\varepsilon} = 1 + \varepsilon \log\left(\frac{\mu^2 4\pi}{a^2}\right) + \mathcal{O}(\varepsilon^2)$

so that

$$\underbrace{\Pi}_{\text{scalar}}(k^2) = -\frac{8e^2}{16\pi^2}\int dx\, x(1-x)\left[\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)\right] \times \left[1 + \varepsilon \log\left(\frac{\mu^2 4\pi}{a^2}\right) + \mathcal{O}(\varepsilon^2)\right]$$

drop $\mathcal{O}(\varepsilon)$ overall

$$\simeq -\frac{2\alpha}{\pi}\int dx\, x(1-x)\left[\underbrace{\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi\right)}_{\equiv 1/\varepsilon'} + \log\frac{\mu^2}{a^2}\right]$$

$\alpha = e^2/(4\pi)$

$$= -\frac{2\alpha}{\pi}\left[\underbrace{\frac{1}{6}}_{\text{from } \int dx\, x(1-x)}\frac{1}{\varepsilon'} - I(k^2)\right], \text{ where}$$

$$I(k^2) = \int dx\, x(1-x) \log\left\{\frac{1}{\mu^2}\underbrace{\left[m^2 - x(1-x)k^2\right]}_{a^2}\right\}$$

Expand for $k^2 \ll m^2$,

$$= \frac{1}{6} \log m^2/\mu^2 \quad \Big\} \; I(k^2=0),$$
$$+ \mathcal{O}(k^2)$$

using $\int dx\, x(1-x) = \frac{1}{6}$

So, $$\pi(k^2) = \underbrace{\pi(0)}_{-\frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon'} - \log\frac{m^2}{\mu^2}\right)} + k^2\pi'(0)$$

We will return to $I(k^2)$ when calculating running of (effective) QED coupling constant

Again, $\pi_{\mu\nu}(k^2)$ depends on μ & ε: unphysical parameters introduced to isolate divergence, thus should cancel in next steps...

... which is adding counterterm (CT)

to cancel divergence:

$$\mathcal{L}^{(\pi)}_{cT} = -\frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu}$$ (same form as $\mathcal{L}_{classical} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$) suffices,

where $Z_3 = 1 - \frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon'} - \log m^2/\mu^2\right)$ ... since

full propagator (classical + loop + CT)

= (similar to re-summing insertions of $\pi_{\mu\nu}$ done earlier)

$$\frac{-i\left(g_{\mu\nu} - k_\mu k_\nu / k^2\right)}{k^2\left[\underbrace{1}_{\text{classical}} - \underbrace{\pi(k^2)}_{\text{loop}} + \underbrace{(Z_3 - 1)}_{cT}\right]}$$

$$= \frac{-i\left(g_{\mu\nu} - k_\mu k_\nu / k^2\right)}{k^2\left[\underbrace{1 - \frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon'} - \log\frac{m^2}{\mu^2}\right)}_{Z_3} \underbrace{+ \frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon'} - \log\frac{m^2}{\mu^2}\right) - k^2\pi'(0)}_{-\pi(k^2)}\right]}$$

So, divergence $\left(\propto \frac{1}{\varepsilon'}\right)$ and μ-dependence cancels in full propagator for all k^2 (unphysical parameters "disappear")

$$= \frac{-i\left(g_{\mu\nu} - k_\mu k_\nu / k^2\right)}{k^2\left[1 - \pi_R(k^2)\right]},$$

where $\pi_R(k^2) = k^2 \underbrace{\pi'(0)}_{\text{finite, } \mathcal{O}(\alpha)}$ $\left[\begin{array}{c}\pi_R(k^2=0) \\ = 0\end{array}\right]$

renormalized

$\Rightarrow$ massless pole survives ... and for on-shell/physical photon $(k^2 \to 0)$, full propagator = classical...

... due to choice of finite part of coefficient of CT (Z_3): can choose differently ...

... but no such freedom in divergent part of Z_3, i.e., that's fixed as above by requirement of canceling divergence

[Again, in general, finite piece of CT is arbitrary $\Rightarrow$ even if no corresponding tree / classical term, there's no prediction from (loop + CT) ... of course, if there is tree contribution, then it's clear from get-go that can't predict...]

"Bottomline" of $\Pi_{\mu\nu}$ calculation: suppose we never heard of gauge invariance (or WT identity). we perform DIMREG calculation — at 1-loop — to find $(k^2 g_{\mu\nu} - k_\mu k_\nu)$ form (after lot of work!)

$\Rightarrow$ photon massless even at (1-) loop...

... but instead if we use hard cut-off regulator, then "surprised" to find otherwise (photon massive at loop-level)

... so, good to have deeper understanding: DIMREG preserves gauge invariance ... + valid at all loops (otherwise, might "wonder"...)