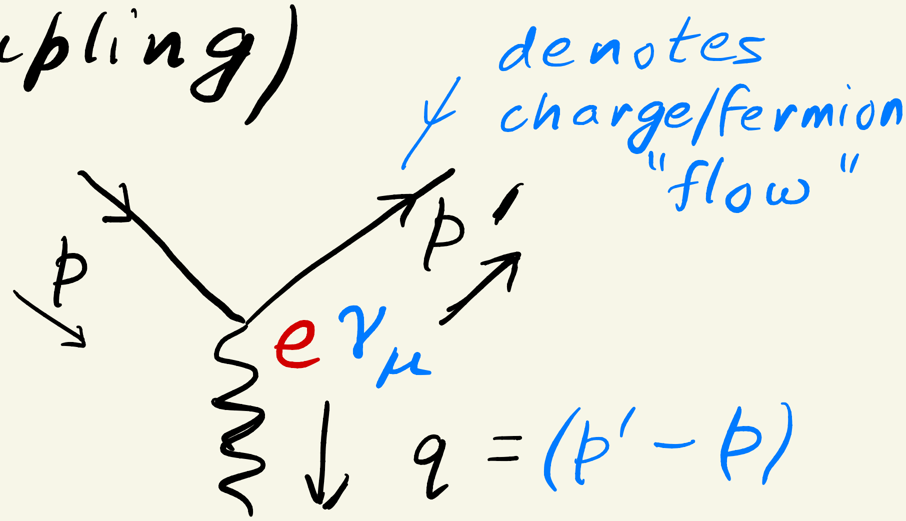
Part (I) (QFT topics)

(A). $\boxed{\text{Renormalizability of QED}}$

– Loop diagrams can be UV-divergent, due to $\int d^4k_{(loop)}$, with $k \to \infty$,

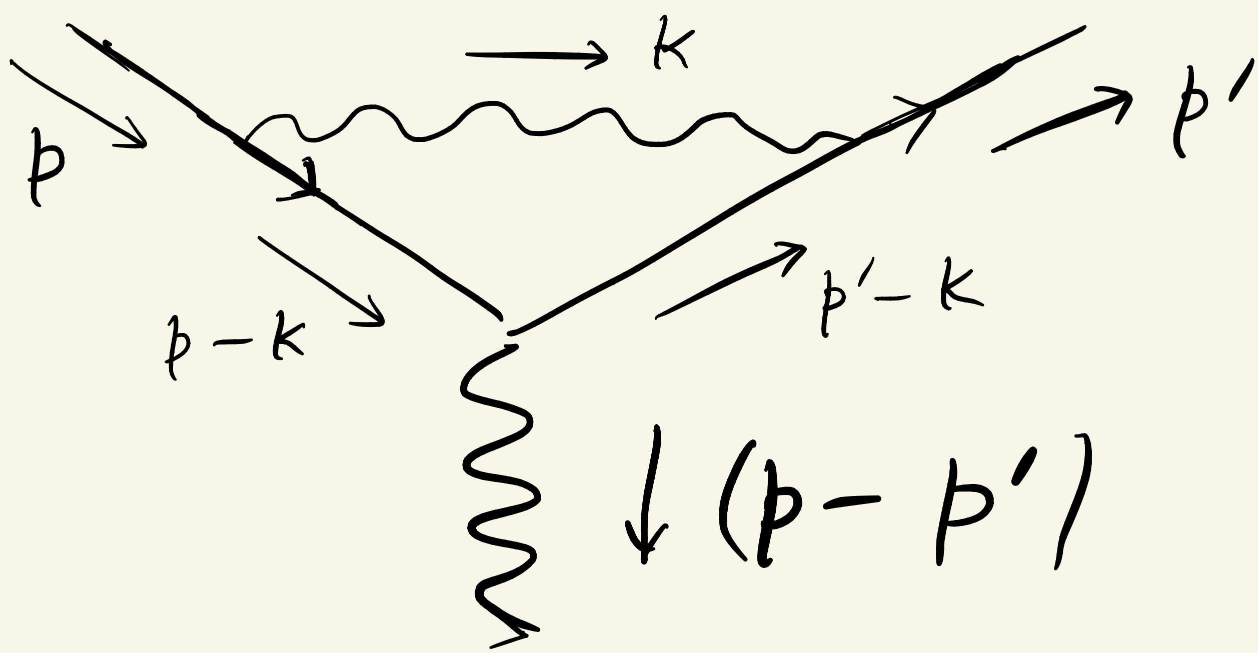
e.g., vertex correction (2 electron-1 photon coupling)

tree-level: $\left[\Gamma^{(0)}\right]$



Loop correction:

$\left[\Gamma^{(1)}\right]$



Schematically (dropping i, γ_μ's : not relevant for divergence structure)

$$\Gamma^{(1)} \sim e^3 \int \frac{d^4k}{(2\pi)^4}$$

loop-momentum integrated over all values

$$\times \frac{g_{\mu\nu}}{k^2}$$

(photon propagator in 't Hooft-Feynman gauge)

$$\left.\begin{array}{l} \times \dfrac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \\ \times \dfrac{(\not{p}' - \not{k}) - m}{(p'-k)^2 - m^2} \end{array}\right\}$$

2 electron/fermion propagators

$$\xrightarrow[k \to \infty]{\text{UV-limit}} \frac{e^3}{16\pi^4} \int d^4k \, \frac{1}{k^2} \frac{\not{k}}{k^2} \frac{\not{k}}{k^2}$$

$$\sim \frac{e^3}{16\pi^4} \int^{\Lambda_{UV}} \frac{d^4k}{k^4} \propto \underline{\log} \Lambda_{UV}$$

Outline of (general) program to "interpret/eliminate" UV divergences

step $\boxed{1}$: Regularization (isolate/parametrize divergence)

- $\int d^4k$ is actually "∞", since must allow k to be arbitrary ($\to \infty$)

- So, introduce a (UV) "regulator", i.e., extra (in the end unphysical) parameter such that integral can be done/is "finite" for certain range of parameter

(Of course, result depends on parameter in such a way that ∞ "returns" if we take physical limit of parameter/remove regulator.)

Concretely, many options: here, mostly choose dimensional regularization (DIMREG): formally, $(4-2\varepsilon)$ number of dimensions such that integral "finite" for $\varepsilon \neq 0$ (but $\to \infty$ for $\varepsilon \to 0$)

– divergent part of amplitude: part which depends on extra parameter and $\to \infty$ in physical limit of parameter.

Step $\boxed{2}$ Add counterterms (CT): product of fields (i.e., "like" terms in Lagrangian) with coefficients chosen so that CT contribution to amplitude cancels

divergent part of same amplitude
(other ways, e.g., possibly in Phys 851)

Step $\boxed{3}$ $\boxed{\text{Full}}$ Lagrangian =
classical (what we started with)
+ (added) CT's (in step 2)
(Roughly speaking: more precise later)
- If CT's of same form as
$\mathcal{L}_{\text{classical}}$, then this process
is just rescaling/renormalization
of original coupling constants &
fields (by divergent constants):
renormalizable theory (e.g. QED)

... seems like "mathematical trick"
to hide infinities (simply
redefinition of pre-existing
terms ?!)

... Not quite: there are observable consequences:

— allows systematic predictions for other amplitudes (again, once divergences are "taken care of")

— "left-over" (finite) effects even from/related to divergent amplitudes, e.g., running of coupling constants

Let's see how it plays out in $\boxed{\text{QED}}$:

already saw 1 example of divergent diagram at 1-loop level (vertex correction) ...

... but what about higher loops? other QED amplitudes also divergent? other theories?

... So, better to have a systematic, general, simple way/formula to know (even if superficial/naive)

$\boxed{\text{degree of divergence of amplitude/diagram } (D)}$

... again, without actually estimating loop integral (which we did for vertex correction), since that's still involved, especially at higher loops!

- We can use formula for D to first (exhaustively) list divergent amplitudes in QED; then apply above program

- Here's summary of results of above process (as "heads-up")
- Formula for D $\Rightarrow$ mass dimension of coupling constants crucial ("natural" units: $\hbar = 1 = c$ ...), e.g.,

$\mathcal{L}_{QED} \sim e\, \bar{\Psi}_e \gamma_\mu A^\mu \psi + \bar{\Psi}_e \not{\partial} \Psi_e + (\partial_\mu A_\nu)^2$

mass dimension of e, denoted by $[e]$ or

$\delta_e = [\mathcal{L}] - 2[\Psi_e] - [A_\mu]$, with

$[S = \int d^4x\, \mathcal{L}] = 0 \Rightarrow [\mathcal{L}] = 4$

$[\bar{\Psi} \not{\partial} \Psi] = [\mathcal{L}] = 4 \Rightarrow [\Psi_e] = 3/2$

$[(\partial_\mu A_\nu)^2] = [\mathcal{L}] = 4 \Rightarrow [A_\mu] = 1$

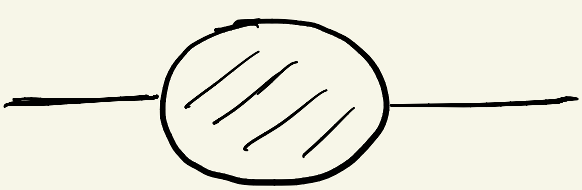
So, $\delta_e = 0$ (dimenisionless)

- If all coupling constants have mass dimension ≥ 0, then only finite number of amplitudes are divergent (at all loop-level)

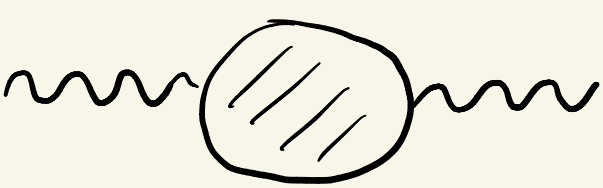
$\Rightarrow$ add CT's for these amplitudes, so cannot predict these (to be taken as input parameter from data) ... but other amplitudes which are finite can be predicted (renormalizable theory)

– However, if any coupling constant has mass dimension < 0, then every amplitude is divergent (at sufficiently high order) $\Rightarrow$ need CT's for all amplitudes, thus no predictions (non-renormalizable)

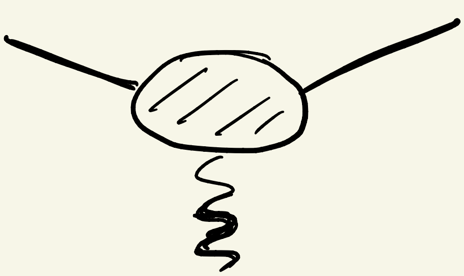
– Apply to QED: $\delta_e = 0$ (coupling constant, e, is dimensionless) $\Rightarrow$ QED is renormalizable $\Rightarrow$ (naively) $\boxed{4}$ (only) amplitudes divergent:



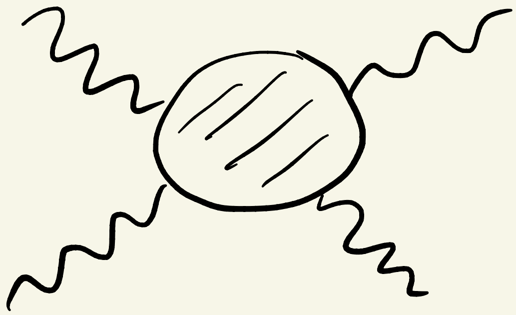
: (naively) linear $\longrightarrow$ (actually) logarithmic



: quadratic $\longrightarrow$ logarithmic



: logarithmic $\longrightarrow$ stays...

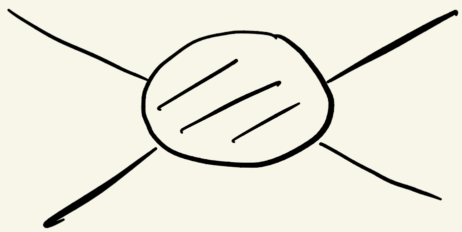
 : logarithmic $\rightarrow$ finite

– Before embarking on (complicated) calculation of these amplitudes, figure out (more easily) their general structure / form using "symmetries" (good to know what to expect)...

... in fact, such analysis will show actual degree of divergence to be smaller than naive/superficial (D) for 3 cases, leaving only 3 divergent

– Furthermore, symmetry will "relate" divergence in 2 of these amplitudes (Ward-Takahashi identity) $\Rightarrow$ absorb divergences in 2 input/free parameters: e & m (fixed by data)

– other amplitudes (finite superficially) can be predicted, e.g.,



(see it quickly using degree of divergence / check it – with bit more effort! – at 1-loop level)

– Onto $\boxed{\text{details}}$...