$\boxed{\text{Non-abelian (global) symmetries: elementary group theory}}$ "motivated" by non-trivial pion-nucleon system (both fields transform)

- Nucleon (N) doublet, made of proton (p) & neutron (n), assumed degenerate ($m_p = m_n \equiv m_N$)

- Pion triplet: ϕ_i ($i = 1,2,3$) or in terms of electric charge eigenstates (see later): $\pi^0 \equiv \phi_3$ & $\pi^\pm \equiv \frac{(\phi_1 \pm i\phi_2)}{\sqrt{2}}$,

real scalars

complex scalar

also assumed to be degenerate: $m^2_{\pi^\pm} = m^2_{\pi^0} = m^2_{\phi_i} \equiv \mu^2$

$$\mathcal{L}_{free} = \bar{\Psi} \underbrace{\left(i\not{\partial} - m_N\right)}_{\propto\, \mathbb{1}_{2\times 2}} \Psi +$$

$$\frac{1}{2}\left(\partial_\mu \Phi\right)^T \left(\partial^\mu \Phi\right) - \frac{1}{2}\mu^2 \Phi^T \Phi, \text{ where}$$

$$\Psi \equiv \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \quad \& \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$

with $\mathcal{L}_{int} = -\frac{1}{4}\lambda\left(\Phi^T \cdot \Phi\right)^4$ } pion self-coupling

$+$ ($\pi - N$ interaction, which is focus here)

$ig\left(\bar{\Psi}\gamma_5 \sigma \Psi\right)\cdot\Phi$ (compactly)

Just to be explicit,

$$\underbrace{\left(\bar{\Psi}\gamma_5 \sigma \Psi\right)}_{\text{"row" in triplet } \phi\text{-space, but number in both Dirac and doublet } N\text{-spaces}}\cdot\Phi = \sum_{a=1,2,3} \left(\bar{\Psi}\gamma_5 \sigma_a \Psi\right)\phi_a$$

$$= \sum_a \left(\psi_p^{+}\gamma_0\gamma_5, \psi_n^{+}\gamma_0\gamma_5\right)\underset{2\times 2}{\sigma_a}\begin{pmatrix}\psi_p \\ \psi_n\end{pmatrix}\phi_a$$

$$= \sum_{\substack{a \\ \text{triplet }(\phi)}} \; \sum_{\substack{ij \\ \text{doublet }(N)}} \; \sum_{\substack{\alpha,\beta \\ \text{Dirac}}} \Psi^{+}_{i\beta}\left(\gamma_0\gamma_5\right)_{\beta\alpha}\left(\sigma_a\right)_{ij}\Psi_{j\alpha}\phi_a$$

... in particular, it contains $\underset{\text{creates } p}{\bar{\psi}_p}\gamma_5\psi_n \underset{\text{annihilates } \pi^+}{\pi^+}$, i.e.,

$p - n - \pi^+$ coupling (γ_5, i.e., pseudo-scalar based on that observation)

— what are (global) symmetries of above $\mathcal{L}$?

(a combination of which is gauged/made local by adding photon field : see later)

(1) (Simple) Baryon-number $[U(1)_B]$, i.e.,

$\Phi \to \Phi$ (π's are mesons) and

$\Psi \to e^{i\theta}\Psi$ ($B = +1$ for both p, n)

(2). (Non-trivial/more interesting) $\boxed{SU(2)}$ $\boxed{\text{on } \Psi}$ and $\boxed{SO(3) \text{ on } \Phi}$:

(a). $SU(2)$ on Ψ (as before) :

$$\Psi \longrightarrow \tilde{U}\,\Psi = \exp\Big(-i\sum_{a=1,2,3}\beta_a \frac{\sigma_a}{2}\Big)\Psi$$

no $\mathbb{1}$, since $SU(2)$

(b). $SO(3)$ [to be explained below] on Φ :

$$\Phi \text{ (column with 3 rows)} \longrightarrow \exp\Big(-i\sum_{a=1,2,3}\beta_a T^a_{adj.}\Big)\Phi,$$

where $T^a_{adj.}$ are 3×3 (hermitian) matrices ("adj." is for adjoint representation : see later):

$$T_1^{adj.} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}; \quad T_2^{adj.} = \begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \& \quad T_3^{adj.} = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note : $T_3^{adj.}$ eigenvectors are $(0\ 0\ 1)^T$, i.e., $\phi_3 (= \pi^0)$ and $(1, \pm i, 0)/\sqrt{2}$, i.e., $\pi^{\pm} \left[= (\phi_1 \pm i \phi_2)/\sqrt{2} \right]$, i.e., 2 of basis vectors $(0\ 1\ 0)$ & $(1\ 0\ 0)$ are not $T_3^{adj.}$ eigenvectors

[Later, we will mention another "basis" for generating same transformation]

- Now, $(i T^a_{adj.})$ are anti-symmetric, real matrices forming a complete set (since these must be purely off-diagonal, of which there are 3 elements) $\Rightarrow R = \exp(-i \beta_a T^a_{adj.})$ are 3×3 orthogonal matrices [sanity check:

$R = \exp A$, where $A = -i \beta_a T^a_{adj.}$ is anti-symmetric so that

$R^T = \exp A^T = \exp(-A) \Rightarrow R^T R = \mathbb{1}$]

And, $\det R = 1$, so called SO(3) transformation (special, orthogonal in 3 dimensions)

[Aside: $\Phi \to -\Phi$ is not SO(3), since transformation matrix $= -\mathbb{1}$ has determinant -1]

– Of course, $R = e^{iB}$ are also unitary, since $B = -\sum_a \beta_a T^a_{adj.}$ are hermitian, but above is not really SU(3) transformation, since $T^a_{adj.}$'s above are (obviously) not complete set of hermitian matrices, which would be $9 = 3^2$ of them, "rest" of those give $[i\, T_a^{rest}]$ as complex, thus cannot "act" on real Φ / i.e., one would need three complex scalars to undergo full SU(3) transformations

- In HW 5.1, check above is symmetry of full Lagrangian (free part and Φ self-coupling is easy, so π–N coupling is the real work here) ... + Noether's theorem...

Actually, are the different "looking" transformations on Ψ (2 dimensional, complex) vs. Φ (real, 3 dimension) related, just different realizations of same symmetry ?!

- Indeed, such a "unified" description might be suggested by number of generators being same for the 2 transformations (i.e., three of σ's and above $T^a_{adj.}$'s) ...

... and (recall Phys 622!) how rotations in $\boxed{\text{real}}$/ physical $\boxed{3\ d}$ are implemented: by 3×3 orthogonal matrices acting on (3d) real vectors ("like" Φ here, but in field space) vs. 2×2 unitary matrices on 2-component spinors (spin-½ system) ("like" Ψ here, but in p, n field space)...

So, we resort to a more systematic approach (mathematical character of these transformations), i.e.,
$\boxed{\text{elementary group theory}}$...
... which will show that above are two representations of <u>same</u>, SU(2) [called $\boxed{\text{isospin}}$] symmetry: we can then generalize this to SU(n)

— Each set of transformation of fields given above forms a "group"

— Group: set of elements/objects with a multiplication operation ($x \circ y$) such that (i). $x \circ y$ belongs to set (for

every choice of x, y) ; (ii) there exists "identity" element : $(x \circ \mathbb{1}) = (\mathbb{1} \circ x) = x$; (iii) "inverse" of every element : $x^{-1} \circ x = x \circ x^{-1} = \mathbb{1}$ and (iv) multiplication is associative :
$$(x \circ y) \circ z = x \circ (y \circ z)$$

— For transformations relevant to physics (e.g. of fields or physical coordinates), "identity" is "no" transformation ; "multiplication" is doing one transformation after another etc.

— Specifically, for (internal) symmetry transformations of fields, we used matrix representations above : we can easily check that $U(n)$ ($n \times n$ unitary) matrices form a group, e.g. $U_1 U_2$ is also unitary ; $[U(n)]^{-1} = [U(n)]^{\dagger}$ ...
... similarly 3×3 orthogonal matrices (R)...

— Transformations of coordinates : Lorentz "rotations" have matrix form, but space-time translations do not quite ($x_\mu \to x_\mu + a_\mu$)

— Here, we will restrict to matrices...

...back to why $SO(3)$ is related to $SU(2)$:

number of group elements in continuous symmetry is ∞, but can be represented by finite number of continuous parameters, e.g., θ for $U(1)$; β_a, where $a = 1, 2 \ldots n^2$ for $U(n)$...

— Generators of group (T^a) are defined by group elements infinitesimally close to $\mathbb{1}$, i.e., $(\mathbb{1} - i \beta_a T^a)$ [β_a small] (T^a is analog of charge for non-abelian group) such that general group element is $\exp(-i \beta_a T^a)$, where β_a need not be small: this holds for $U(n)$, $SO(n)$ ... called $\boxed{\text{Lie}}$ groups

— generators must satisfy certain properties in order to reproduce correct group multiplication, e.g., we can be motivated by generators of $SU(2)$, i.e., $\sigma^a/2$ which form a closed algebra under commutation relations : $\left[\sigma_{a/2}, \sigma_{b/2}\right] = i\varepsilon_{abc}\left(\sigma_{c/2}\right)$

– Indeed, in general, if

$$\left[T_a, T^b\right] = i f_{abc} T_c ,$$

then $\exp(-i\beta_a T^a)$ form a group, since $e^A e^B$ (where A, B are of $-i\beta_a T^a$ form)

= (using Baker-Campbell-Hausdorff formula)

$$\exp\left\{A + B + \frac{1}{2}\underbrace{[A,B]}_{C} + \frac{1}{12}\left(\underbrace{\left[A,\underbrace{[A,B]}_{C}\right]}_{D} - \underbrace{\left[B,\underbrace{[A,B]}_{C}\right]}_{E}\right) + \dots\right\}$$

$$= \exp\left(A + B + \dots C + \dots D + \dots E + \dots\right)$$

(where C, D, E are same form as A, B due to commutation relations of T^a's)

= same form as e^A, e^B, i.e.,

$$\exp\left[-i\beta_a^{(1)} T^a\right] \times \exp\left[-i\beta_b^{(2)} T^b\right] = \exp\left[-i\beta_c^{(3)} T^c\right]$$

– Thus, group is "defined" by commutation relations of its generators (Lie algebra), e.g., (1). SU(2) acting on doublet Ψ : $T^a = \frac{\sigma^a}{2}$ $(a=1,2,3)$

with $[\sigma_a/2, \sigma_b/2] = i\varepsilon^{abc}(\sigma_c/2)$

- Clearly f_{abc} (called $\boxed{\text{structure constants}}$) $= -f_{bac}$, i.e.,
antisymmetric in a, b (following from commutation relations:

$$[T_b, T_a] = -[T_a, T_b] = -if_{abc}T^c$$
$$= +if_{bac}T^c$$

- Furthermore, T^a's can be chosen such that f_{abc} is totally anti-symmetric: $f_{abc} = -f_{acb}$
$= -f_{cba}$;

e.g., ε_{abc} for SU(2) above

- Remarkably,

(2). SO(3) on (real) triplet Φ has

Lie algebra identical to $SU(2)$:

$$\left[T^a_{adj}, T^b_{adj}\right] = i\varepsilon_{abc} T^c_{adj}$$

i.e., $\left[T^a, T^b\right] = i\varepsilon_{abc} T^c$ for

$T_a = \sigma_a/2$ (2 x 2 matrices: traceless & hermitian)

& $T_a^{adj.}$, where $i T_a^{adj.}$ are real, antisymmetric (3×3) matrices

$\Rightarrow$ elements of group generated by $T_a^{adj.}$ (i.e., 3 x 3 orthogonal matrices, with determinant 1) are in one-to-one correspondence with those generated by $\sigma_a/2$ (2 x 2 unitary matrices, with determinant 1). Namely, if we have

$$\underbrace{\exp\left[-i\beta_a^{(1)}\frac{\sigma_a}{2}\right] \times \exp\left[-i\beta_b^{(2)}\frac{\sigma_b}{2}\right]}_{\text{product of 2 group elements: 2 x 2 unitary matrices}} = \underbrace{\exp\left[-i\beta_c^{(3)}\frac{\sigma_c}{2}\right]}_{\text{another group element}},$$

then (i.e., necessarily) we have for 3 x 3 orthogonal matrices,

$$\exp\left[-i\beta_a^{(1)} T_a^{adj}\right] \times \exp\left[-i\beta_b^{(1)} T_b^{adj}\right] = \exp\left[-i\beta_c^{(3)} T_c^{adj}\right]$$

[i.e., same 3 β_a's for both group element multiplications]

– we say SU(2) is isomorphic to SO(3) or these are simply 2 different representations of SU(2): doublet/fundamental and triplet/adjoint ...

$\Rightarrow$ $\pi - N$ system has just one non-abelian symmetry, i.e., SU(2): it's just that π, N transform as different representations under it (again doubet for Ψ vs. triplet for Φ)

– we will generalize above 2 representations to SU(n), but before that, a couple of more comments on the example above:

— we can gauge a subgroup of global symmetry of $\pi - N$ system, i.e., $SU(2)_{isospin} \times U(1)_B$, by coupling to EM field: see separate note [later on, we will find something similar with the full electro-weak gauge group, i.e., $U(1)_{EM}$ is a sub-group of $SU(2)_L \times U(1)_Y$]

— $\boxed{\text{Another}}$ basis for generators of adjoint (or triplet) representation of $SU(2)$ is given by (see CL p. 93):

$$J_3 \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad J_1 \propto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\&\; J_2 \propto \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & -i \\ 0 & +i & 0 \end{pmatrix}$$

$\Rightarrow$ basis vectors $(1\ 0\ 0)^T$, $(0\ 1\ 0)^T$ & $(0\ 0\ 1)^T$ are eigenvectors of J_3 with eigenvalues $+1, 0$ & -1 (respectively), with $J_{\pm} = \dfrac{J_1 \pm i J_2}{\sqrt{2}}$ being raising/lowering operators for J_3 eigenvalue:

$$J_+ \propto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \& \quad J_- \propto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(One can check that $[J_a, J_b] = i\,\varepsilon_{abc} J_c$, as is familiar from angular momentum theory)

- In this basis $\pi^0 \sim (0\ 1\ 0)^T$ (T_3 eigenvalue 0), while $(1\ 0\ 0)^T$ & $(0\ 0\ 1)^T$ are $\pi^\pm$, with T_3 eigenvalues ± 1 (respectively)

... cf. earlier basis, i.e., generators $T_a^{adj.}$ being (imaginary) anti-symmetric 3×3 matrices so that group elements $\exp[-i\beta_a T^a_{adj.}]$ are orthogonal (real) matrices; $T^3_{adj.}$ eigenvectors are $(0\ 0\ 1)^T$ & $(1\ \pm i\ 0)^T/\sqrt{2}$ with eigenvalues 0 (i.e., π^0) & ± 1 (i.e., $\pi^\pm$) (respectively)

- The 2 bases are related by a "similarity transformation": this could be useful in HW (e.g. - in a different context - is in Eq. 4.72 of CL)

- Onto $SU(n)$, where

(1). $\boxed{\text{fundamental}}$ (n-dimensional) representation was already studied earlier: generators, $\boxed{T^a_{fund}}$, are $(n\times n)$ traceless hermitian matrices, group elements are $(n \times n)$ unitary matrices with determinant 1 ... acting on (complex) column vectors with n rows

– these T^a_{fund} can give us structure constants of $SU(n)$, i.e.,

$$\left[T^a_{fund}, T^b_{fund} \right] = i f_{abc} T^c_{fund}$$, where

f_{abc} can be chosen to be totally anti-symmetric

(2) $\boxed{\text{adjoint}}$ representation: we can be "inspired" here by $SU(2)$, i.e., dimension of this representation [$= 3$ for $SU(2)$] is same as number of generators (which of course is same for all representations of a group) ... and $T^a_{adj.}$ can actually be "formed" out of structure constants: indeed, for $SU(2)$, we see that $\left(T^a_{adj.}\right)_{bc} = i \varepsilon_{abc}$

– So, in general, we can show

$$\left(T^c_{adj.}\,[\text{for } SU(n)]\right)_{\underbrace{ab}_{\substack{(n^2-1)\times(n^2-1) \\ \text{matrix (dimension} \\ \text{of representation)}}}} = -i f_{abc}$$

c: $1 \ldots (n^2-1)$ (number of generators)

i.e., that above $T^a_{adj.}$ satisfy $SU(n)$ Lie algebra: see HW 5.2.1

... of course, we have "trivial" representation for any group (called $\boxed{\text{singlet}}$), i.e., $T^a = 0$ ... so, only group element is $\mathbb{1}$ (no transformation)

& other representations (matrices acting on column vectors of other dimensions), but we won't need beyond above 3 here

[In general, representation of group, i.e., elements, need not be matrices and "multiplication" might actually be "addition", e.g., integers: $\mathbb{1}$ is "0", inverse of integer is its negative ...]