Non-abelian symmetries: $\boxed{\text{global}}$

Motivation (reminder!): in order to transition from QED [massless, $U(1)$, gauge boson] to full SM, we need two extensions/generalizations:

(i) renormalizable theory of massive gauge boson [for weak (nuclear) force, with short range vs. long in QED]: (gauged) SSB / Higgs mechanism (2nd QFT topic) and

(ii). Non-abelian gauge theory: needed to describe (a) strong (nuclear) force, where coupling binding partons (constituents of hadrons) becomes weaker at higher energies (cf. IR-free in QED); due to self-interactions of gauge bosons in vacuum polarization ... and

(b) weak (nuclear) force: W^+ couples μ^-, e^- to ν_μ, ν_e (i.e., off-diagonally vs. QED)...

... again, independent of above "applications", non-abelian gauge (Yang-Mills) theory is a robust, fascinating possibility within QFT...

– So, 3^{rd} QFT topic is to first generalize global, internal symmetries from $U(1)$ to $SU(n)$. note internal implies transformation involves fields at same space-time point (i.e., do not touch x_μ or Dirac index of ψ_α ...) vs space-time symmetries connect fields at different space-time points (mix-up Dirac index ...), e.g., Lorentz transformation; global stands for transformation parameters being space-time independent ...

... then gauge these "new" symmetries (promote parameters to be space-time derivatives, coupling gauge fields to matter...)

Outline for non-abelian global symmetries:

– warm-up: from $U(1)$ to $SU(n)$ ...

– elementary group theory [including $SO(n)$...]

– SSB of non-abelian (global) symmetries (NGB's ...): a bit more complicated than $U(1)$, e.g., part of symmetry unbroken (use multiple scalar field discussion)

From $U(1)$ to $U(2)\ldots SU(n)$

— Symmetry/invariance under global phase rotation (including interactions, even if not gauge), e.g.,

(1). $\mathcal{L}_{\psi,\phi}$ $\leftarrow$ real $= \bar{\psi}(i\not{\partial}-m)\psi + \partial_\mu\phi\partial^\mu\phi + \underbrace{\bar{\psi}\psi\phi h}_{\text{Yukawa coupling}}$

is invariant under

$\phi \to \phi$ (unchanged); $\psi_{\alpha=1\ldots4} \to \psi_\alpha e^{i\theta}$ $\leftarrow$ spacetime-independent (since global)

($\alpha = 1\ldots4$: Dirac index)

α: untouched since internal symmetry

(2). $\mathcal{L}_{\Phi}$ $\leftarrow$ complex $= (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$,

λ: quartic (self-)coupling

with $\Phi \longrightarrow e^{i\theta}\Phi$

... called abelian or $U(1)$ symmetry, since transformation, $e^{i\theta}$, is 1×1 "matrix", which is unitary, i.e., $(e^{i\theta})^\dagger(e^{i\theta}) = 1$:
∞ number of transformations, parametrized by single (continuous, real) parameter (θ)

— Aside: two different matrix spaces/structures: internal (trivial above, since only 1 ψ, cf. below) vs. Dirac space (which is non-trivial above, but not focus here, i.e., for internal symmetry it goes along for ride as above): anyway, for sake of completeness,

$(\bar{\Psi})_\beta$ ("row") $\equiv (\Psi^+)_\alpha (\gamma_0)_{\alpha\beta}$ (Ψ_β is column);

$\bar{\Psi}\gamma^\mu\Psi = (\Psi^+)_\alpha (\gamma_0)_{\alpha\beta} (\gamma^\mu)_{\beta\delta} \Psi_\delta$ etc...

(not shown explicitly in above $\mathcal{L}$ ...)

$\Rightarrow$ these 2 spaces/structures are "independent/commuting"

— Onto 2 fermions (free for now) with different masses (in general):

$$\mathcal{L}_{\psi_i} = \sum_{i=1,2} \bar{\Psi}_i (i\partial\!\!\!/ - m_i)\Psi$$

— notation: "upper case" Ψ for "doublet" (in internal space): $\Psi_{(\alpha)} = \begin{bmatrix} (\psi_1)_{(\alpha)} \\ (\psi_2)_{(\alpha)} \end{bmatrix}$

so that suppressing Dirac structure:

$$\mathcal{L}_{\Psi} = \bar{\Psi}\left[i\partial\!\!\!/ \underbrace{\mathbb{1}_{2\times 2}}_{\text{internal/doublet space}} - \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\right]\Psi$$

– symmetry is $U(1)_1 \times U(1)_2$, i.e., separate phase rotations on $\psi_{1,2}$

Special case: $m_1 = m_2$ so that ("dropping" double-space structure also):

$$\mathcal{L}_\Psi = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi, \text{ where (again)}$$

$$\bar{\Psi}\Psi = (\bar{\psi}_1 \;\; \bar{\psi}_2)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = (\psi_1^\dagger\gamma_0, \underbrace{\psi_2^\dagger\gamma_0}_{\text{row in Dirac space (but "suppressed" here)}})\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$= \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$$

$$\left(= \psi_1^\dagger\gamma_0\psi_1 + \psi_2^\dagger\gamma_0\psi_2\right)$$

= (spelled out, including summation over repeated indices) $\underbrace{\sum_i}_{\text{doublet/internal space}} \underbrace{\sum_\alpha \sum_\beta}_{\text{Dirac space}} \Psi_{i\alpha}^\dagger (\gamma_0)_{\alpha\beta} \Psi_{i\beta}$

– Above $\mathcal{L}$ invariant mixing $(\psi_1)_\alpha$ with $(\psi_2)_\alpha$: "1,2" here refer to doublet/internal space, i.e, rotation in doublet/internal space, with Dirac index α "staying put", cf. Lorentz (space-time vs. internal above) transformation rotates $(\Psi_1)_\alpha$ into $(\Psi_1)_\beta$ (i.e., does not touch doublet space: $\Psi_{1,2}$ transform independently)

– Mathematically, $\mathcal{L}_\Psi$ invariant under $U(2)$ [2 × 2 unitary matrix] transformation:

$$\Psi' (\text{new}) = U_{2\times 2} \Psi, \quad U^\dagger U = \mathbb{1}_{2\times 2}$$

– In general, with n ψ's of same mass, we have $U(n)$ symmetry, with

$$U_{n\times n} = \exp\left[-i \sum_a \beta_a \left(T^a_{n\times n}\right)\right], \text{ where}$$

$$\exp(A\text{-matrix}) \equiv \mathbb{1} + A + A^2/2! + \dots \frac{A^n}{n!} + \dots;$$

β_a are real parameters and $T^a_{n\times n}$ are hermitian, so that $U^\dagger = \exp(+i\beta_a T^a)$, giving $U^\dagger U = \mathbb{1}$

– Now, $(n \times n)$ complex matrix has $2n^2$ real parameters, but hermitian condition reduces that to n^2, since diagonal elements of T^a are real, while (appropriate) off-diagonal elements are complex conjugates of each other

$\Rightarrow$ $a = 1, 2 \ldots n^2$ (T^a's are called generators of this fundamental representation: more on this group theory later, including other representations)

— e.g., for $n = 2$, we get $T^a = \mathbb{1}, \vec{\sigma}$, where $\vec{\sigma} = \sigma_{1,2,3}$ are Pauli matrices

— In general, choose $T^{a=n^2} = \mathbb{1}_{n\times n}$, so that other (all independent) T^a's, $a = 1, 2 \ldots (n^2 - 1)$ are traceless

$$\Rightarrow U_{n\times n} = e^{-i\beta_{n^2}\mathbb{1}} \times \tilde{U}_{n\times n}, \text{ where}$$

$$\tilde{U}_{n\times n} = \exp\Big[-i \sum_{a=1}^{n^2-1} \beta_a \underbrace{T^a}_{\text{traceless}}\Big]$$

— Using (schematically) $\det e^A \sim e^{\operatorname{tr} A}$, we see that $\tilde{U}$ have determinant $\mathbb{1}$, so this subgroup of transformations are called $SU(n)$, where "S" denotes special / determinant $= 1$

- Clearly, the 1st part of U, i.e., $\exp(-i\,\beta_{n^2})\,\mathbb{1}$ is a $U(1)$ transformation [multiplies all ψ's by same phase]
- So, schematically, $\boxed{U(n) = SU(n) \times U(1)}$

[e.g., $SU(2)$ "generated" by $\sigma_{1,2,3}$]

- Even more special case with n ψ's: m (common) $= 0$ $\Rightarrow$ symmetry enlarged to $U(n)_L \times U(n)_R$, i.e., separate rotations on L, R chiralities (since only mass term "connects" them)
- We can also include (rather trivially) interactions which are $SU(n)$ invariant, e.g.

(1). $\mathcal{L}_{Yukawa} = \sum_{i,j} h_{ij} \overline{\psi}_i \psi_j \phi = \overline{\Psi}\, h\, \Psi\, \phi$ → matrix

where ϕ is real scalar, not transforming under $U(n)$: for arbitrary h_{ij}, this coupling is not $U(n)$-invariant, but it is $U(n)$ invariant if we choose $h_{ij} = h\,\delta_{ij}$ $(h \propto \mathbb{1})$

(2). All n ψ's have electric charge $+1$ (couple identically to photon): $\partial\!\!\!/\, \mathbb{1}$ of earlier $\rightarrow (\partial\!\!\!/ + ie 1)\,\mathbb{1}$, i.e., $SU(n)$ is still global symmetry, but $U(1)$ part of $U(n)$ is gauged (local symmetry), with A_μ (photon) being $SU(n)$-invariant (like ϕ), but of course A_μ transforms (inhomogeneously) under local $U(1)$ ...

... onto more non-trivial [$U(n)$-invariant] interactions, i.e., where ψ's couple to a field which is not a "singlet": it transforms under this symmetry