Spontaneous breaking of non-abelian (global) symmetries: SU(2) examples

- main point: similar to U(1), but richer, i.e., multiple possibilities for pattern of breaking, since start with > 1 symmetry generators

SSB of SU(2) by (real) scalar triplet VEV (more in HW 5.3)

- Consider (pure) pion part of $\pi - N$ Lagrangian of before (obviously, fermions - nucleons here - cannot get VEVs!):

$$\mathcal{L}_{\Phi} = \frac{1}{2}(\partial^\mu \Phi)(\partial_\mu \Phi) - \frac{1}{2}\mu^2 \Phi^T \Phi - \frac{1}{4}\lambda(\Phi^T \Phi)^2$$

where $\Phi = (\phi_1 \; \phi_2 \; \phi_3)^T$ is real triplet [adjoint representation of SU(2)] scalar field

- choose $\mu^2 < 0$ such that $\langle 0 | \underbrace{\Phi^T \Phi}_{\phi_1^2 + \phi_2^2 + \phi_3^2} | 0 \rangle = v^2 = -\frac{\mu^2}{\lambda}$

- Just like U(1) case, where $\langle 0 | (\phi_1 + i\phi_2)/\sqrt{2} | 0 \rangle = \frac{v}{\sqrt{2}}$, vacuum in SU(2) model is not unique: let us choose (without loss of generality), only $\langle 0|\phi_3|0\rangle \neq 0$, i.e., $\langle 0|\underset{\uparrow}{\Phi}|0\rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

column-vector

– Plug above VEV into $\mathcal{L}_\Phi$ (see HW 5.3.1) to find $\phi_{1,2}$ are massless, while (fluctuation around VEV of) ϕ_3 is massive...

... we sort of expected to find massless scalars as NGB's due to SSB, i.e., $\phi_{1,2}$ are (strong) "candidates" for being NGB's

– In fact, Goldstone's theorem says there is a massless scalar (NGB) corresponding to each broken symmetry generator ⇒ in this model, at most 2 generators are broken ... out of 3 $(= 2^2 - 1)$ to begin with for SU(2) ⇒ at least one symmetry is unbroken in vacuum...

... which was "expected" as follows: SU(2) here is really SO(3) acting on Φ, i.e.,

"rotations" among ϕ_a ($a = 1, 2, 3$) $\Rightarrow$
$\langle 0 | \phi_3 | 0 \rangle$ breaks this rotational
symmetry by "picking a direction in 3d"...
but 2 d rotations, i.e., among $\phi_{1,2}$, are
still a symmetry: this is SO(2)
rotating 2 real scalar fields into each other
or $\boxed{U(1)}$ on 1 complex scalar field

— So, we have
SU(2) [3 generators] $\rightarrow$ U(1) [1 generator], i.e.,
(only) 2 generators are broken
(out of original 3) $\Rightarrow$ 2 NGB's...
... which then "must be" $\phi_{1,2}$

— Check explicitly unbroken U(1)
in HW 5.3.1; another way to
"see" unbroken symmetry in
HW 5.3.2; "identity" of NGB
matches Goldstone's theorem HW 5.3.3

- Rank of group is number of diagonal generators in fundamental representation, i.e., 1 for SU(2) & U(1)

$\Rightarrow$ In general, rank of group is not changed/reduced if SSB by adjoint VEV [here, triplet for SU(2)]

SSB of SU(2) by VEV of (complex) doublet scalar field [cannot have real doublet, since (complex) unitary rotations involved... triplet can be real, since real, orthogonal rotations]

- Subtlety (see HW 5.4): 3 massless scalars (NGB's), hence 3 broken generators, i.e., all original SU(2)...

... but starting symmetry (number of generators) is actually larger than 3 (again, 3 broken) $\Rightarrow$ there's remnant symmetry!

... in general, rank reduced by VEV of fundamental scalar