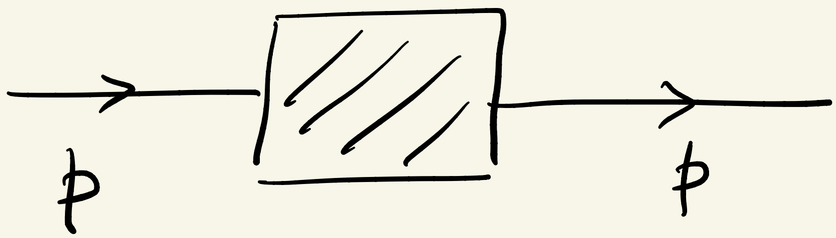
Fermion self energy (Σ)

- Summary: recall



has (naive/superficial) $D = 1$ (linear divergence), but "symmetry" lowers it to $D = 0$ (logarithmic): general argument (valid for all loops) + check by explicit calculation at 1-loop (but no need to use specific regulator, cf. $\Pi_{\mu\nu}$, where DIMREG was different than hard cut-off...)

- General argument (based on structure in Dirac space, "chiral" symmetry and Lorentz invariance):

– Note that Σ (appearing in-between two external fermions, even if off-shell) is (4×4) matrix in Dirac space $\Rightarrow \Sigma$ must be a combination of $\mathbb{1}$, γ_μ, $\sigma_{\mu\nu}$, γ_5 & $\gamma_\mu \gamma_5$ (in general)

– In short, QED is parity-invariant (no γ_5 in interaction) $\Rightarrow \Sigma$ also can't contain γ_5 or $\gamma_\mu \gamma_5$ ... so, left with $\mathbb{1}$, γ_μ & $\sigma_{\mu\nu}$

– Now, Σ is Lorentz scalar : so, must have $\sigma_{\mu\nu} p^\mu p^\nu$ (only 4-vector available is p^μ) $= 0$ (by anti symmetry of $\sigma_{\mu\nu}$)

– Similarly, $\gamma_\mu p^\mu \ldots \Rightarrow$

$$\boxed{\Sigma(p) = a(p^2)\, \not{p} + b(p^2)\, \mathbb{1}}$$

$\Rightarrow$ degree of divergence of $a(p^2)$ $= 0$ (lower than superficial $= 1$ due to presence of external momentum (as per general arguments)

- Onto, $\Sigma \ni b(p^2)\,\mathbb{1}$: "connects" ψ_L to ψ_R, i.e., "violates" chiral symmetry : $\psi_L \longrightarrow e^{+i\alpha}\psi_L$ vs. $\psi_R \rightarrow e^{-i\alpha}\psi_R$ (i.e., opposite transformation for ψ_L vs. ψ_R) vs. EM/gauge symmetry :

$$\psi_{L,R} \rightarrow e^{-ieQ\theta(x)}\psi_{L,R}$$ (same transformation for L, R)

- Is there any term in $\mathcal{L}_{\text{classical}}$ which violates chiral symmetry? Yes, it's $m\bar{\psi}\psi$

$\Rightarrow b(p^2)$ must be $\propto m$

(Again, if $m \to 0$, then $\mathcal{L}_{\text{classical}}$ has chiral symmetry; so will Σ, hence $b(p^2)$ must vanish in this limit)

$\Rightarrow$ actual degree of divergence of $b(p^2)$ is also lower by 1 than naive, i.e., $b(p^2)$ is at most log-divergent [we must obtain "m" in $b(p^2)$ by expanding fermion propagator $(k \gg p, m) \sim 1/k + p/k^2 + \frac{m}{k^2}$ $\Rightarrow$ 1 more power of k in denominator]

[More "carefully," use dimensional analysis.

$[\Sigma] = 1$ (again, roughly "corresponds to" $\bar{\psi}\Sigma\psi$ "term" in Lagrangian)

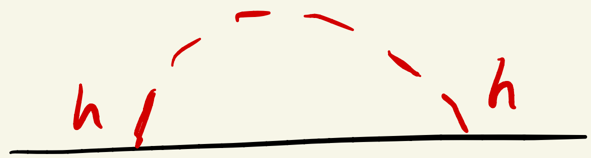
not possible

$\Rightarrow [b(p^2)] = 1$ and $b(p^2) \propto m$,

but then $b(p^2) \sim m \boxed{\Lambda^{>0}} \times$(couplings) $\Rightarrow$ no power/linear divergence (if [couplings] ≥ 0, i.e., renormalizable theory)

but $b(p^2) \sim m \log \Lambda$ is allowed: indeed will find this at 1-loop ...]

– Note above arguments do not make use of gauge invariance, so are valid for (pure) Yukawa theory as well, i.e.,

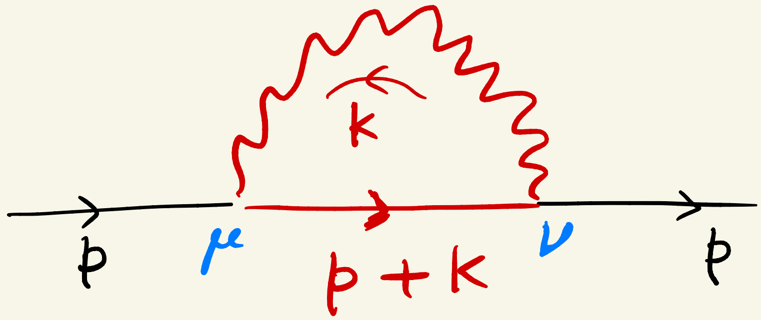


[Relatedly, we don't need to be so "careful" with DIMREG vs. hard cut-off here ...

... cf. lowering of degree of divergence of $\Pi_{\mu\nu}$ (photon self-energy) relied on gauge invariance/WT identity, so had to use regulator respecting gauge invariance, i.e., DIMREG, to make it work ... + self-energy of scalar in Yukawa theory not "protected" analogously, i.e., $D=2$ remains ...]

– Above is valid at all loops: onto more explicit results at one loop

– As usual, schematically first, which will already suggest above structure (even if we did<u>n</u>'t know "symmetry" arguments)



$$\Sigma \sim \int d^4k \; \frac{1}{k^2} \underbrace{\frac{1}{(\not p + \not k - m)}}_{\text{expand for } k \gg p,\, m}$$

(neglect Dirac structure)

$$\sim \int d^4k \; \frac{1}{k^2} \left(\frac{1}{k} + \frac{p}{k^2} + \frac{m}{k^2} + \dots \right)$$

linear divergence, but odd powers of k in integrand $\Rightarrow \int d^4k$ expected to vanish

log-divergence in $a(p^2)$

log-divergence in $b(p^2)$

(cf. $\Pi_{\mu\nu}$ where do not see — at this level — how $D=2$ — naive quadratic divergence — will be lowered: really have to calculate in detail)

— will check above educated guess explicitly

— Motivation: (i) (again) trust, but verify! No further reduction of divergence, i.e., beyond schematic / estimate above (again, no "extra" principle, cf. $\Pi_{\mu\nu}$, where quadratic divergence of estimate vanishes in detail: of course, expected from WT identity, but again not seen schematically ...);

(ii) more practice;

(iii). relation between (divergence in) Σ and Γ_μ, based on gauge invariance (or WT identity): we do need to use DIMREG in order to obtain this, even if we can see above structure of Σ by itself without "proper" regularization

— Explicit calculation of $\Sigma(p)$ at 1-loop: details of DIMREG in HW 2.1

[Pauli-Villars (PV) regulator, which also preserves gauge invariance, done in LP sec. 12.6.2]

- Focus on getting $a(p^2)\cancel{\not p} + b(p^2)$ structure

$\propto m$

- Using 't Hooft-Feynman gauge for photon propagator, we get (set $Q=-1$)

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{(\not p + \not k + m)}{[(p+k)^2 - m^2]} \gamma^\mu \frac{1}{k^2}$$

step 0

p^2 need not be m^2

also μ^ε in DIMREG

due to $g_{\mu\nu}$ from photon propagator

step $\boxed{1}$: Use $\gamma^\mu \gamma_\mu = 4 \, \mathbb{1}$ $\longrightarrow (4 - 2\varepsilon)$ in DIMREG

and $\gamma_\nu \gamma_\mu \gamma^\nu = -2\gamma_\mu$ $\longrightarrow (-2+\varepsilon)$ in DIMREG

("instead" of traces in $\Pi_{\mu\nu}$)

$$\Sigma(p) = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2(\not p + \not k) - 4m}{[(p+k)^2 - m^2]\, k^2}$$

step 2 : Combine denominators using Feynman parameters

$$= ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{2(\not{p}+\not{k})-4m}{\left\{x\left[(p+k)^2-m^2\right]+k^2(1-x)\right\}^2}$$

$$= ie^2 \int dx \int \frac{d^4k}{(2\pi)^4} \frac{2(\not{p}+\not{k})-4m}{\left[k^2+2xk\cdot p+x(p^2-m^2)\right]^2}$$

step $\boxed{3}$: "complete square" in denominator

$(k' = k + px)$

$$= ie^2 \int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{\left[2\not{k}'+2\not{p}(1-x)-4m\right]}{\left[k'^2+x(1-x)p^2-m^2x\right]^2}$$

Since denominator has even powers of k', the 1st term (which has naive, linear divergence), having odd power of k' in numerator, vanishes upon $\int d^4k'$ (more clear upon Wick rotation), as was guessed schematically earlier.

whereas, 2^{nd} & 3^{rd} terms "match"

$a(p^2)$ & $b(p^2)$ of above expectation

(again, with log divergence $\sim \int d^4k' / [k'^2]^2$)

... see HW 2.1 for DIMREG;

LP sec. 12.6.2 for PV ...

Counterterms for Σ

Since $\Sigma(\not{p}) = a(p^2)\, \not{p} + b(p^2)$,

log-divergent (all loops)

we can choose

$$\mathcal{L}^{(\Sigma)}_{CT} = (Z_2 - 1)\, \bar{\psi} \left[i \not{\partial} - (m - \delta m) \right] \psi ,$$

where Z_2, δm are independent of p

[again, same form as $\mathcal{L}_{classical} = \bar{\psi} (i \not{\partial} - m) \psi$]

– Note, need two coefficients of CT to cancel independent divergences in a, b

$\Rightarrow$ full fermion propagator (classical + loop + CT):

$= -i / [\not{p} - m - \Sigma_R(\not{p})]$, where

$$\Sigma_R(p) \equiv \underbrace{\Sigma(p)}_{\text{loop}} + \Sigma_{CT}(p), \text{ with}$$

$$\Sigma_{CT}(p) \equiv a_{CT}\not{p} + b_{CT}, \text{ where}$$

$$\left.\begin{aligned} a_{CT} &= -(Z_2 - 1) \\ b_{CT} &= (Z_2 - 1)(m - \delta m) \end{aligned}\right\} \text{ no } p \text{ dependence here}$$

$\boxed{\text{Choose}}$ a_{CT}, b_{CT} (really finite parts) such that for $\boxed{\not{p} \to m}$ (on-shell fermion), full propagator $\longrightarrow$ classical : $-i/(\not{p} - m)$, i.e.,

$$\Sigma_R(p)\Big|_{\not{p} \to m} \text{ must be } \sim \mathcal{O}(\not{p} - m)^2$$

[no $(\not{p} - m)^{0 \text{ or } 1}$] so that full propagator $= \dfrac{-i}{\not{p} - m} \dfrac{1}{[1 + \mathcal{O}(\not{p} - m)]}$

(similar to choice of Z_3 for photon giving full propagator $\sim \dfrac{-i}{k^2[1 + \mathcal{O}(k^2)]}$

$\rightarrow$ classical $\sim 1/k^2$ as $k^2 \to 0$)

$\Rightarrow a_{CT}, b_{CT}$ obtained by requiring

(more in HW1.)

$$\Sigma_R \big|_{\not{p}=m} = 0 \quad \& \quad \underbrace{\frac{\partial}{\partial \not{p}} [\Sigma_R(\not{p})] \Big|_{\not{p}=m}}_{\text{coefficient of } (\not{p}-m)^1 \text{ in expansion}} = 0$$

[meaning of $\boxed{\partial/\partial\not{p}}$ acting on $\Sigma_R(\not{p})$:

just write p^2 contained in $\Sigma(p)$ as $(\not{p}\,\not{p})$, so Σ_R can be thought of as function of $\not{p}$ ...]

$$\Rightarrow a_{CT} = -a(m^2) - 2m^2 a'(m^2) - 2m\, b'(m^2)$$

$$\left[a' = \frac{\partial}{\partial p^2} a(p^2) \ldots\right]$$

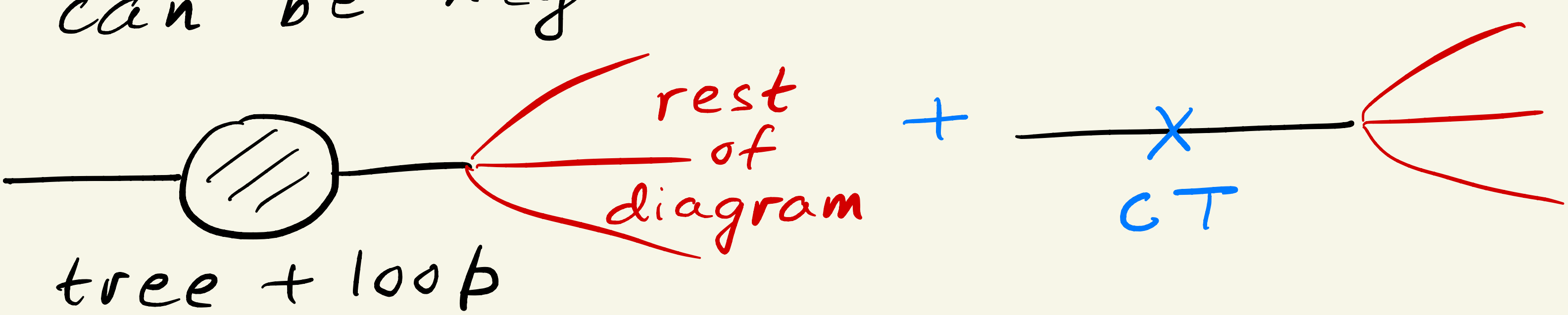
and $$b_{CT} = -b(m^2) + 2m^3 a'(m^2) + 2m^2 b'(m^2)$$

— Check that Σ_R is finite even for $\boxed{\not p \neq m}$ (off-shell fermion):

$$a(p^2) = a(m^2) + a'(m^2)(p^2 - m^2) + \dots$$
$$b(p^2) = b(m^2) + b'(m^2)(p^2 - m^2) + \dots$$

The point is that $\partial/\partial p^2$ lowers degree of divergence, i.e., $a'(m^2)$, $b'(m^2)$ are finite; divergences contained in $a(m^2)$, $b(m^2)$ and it's easy to see that $a(m^2)$, $b(m^2)$ cancel between Σ (loop) & Σ_{CT} in Σ_R

— Also, with above choice of CT, (divergent) "dressing" of external fermion in on-shell/physical limit can be neglected:



where loop cancels CT ($\not{p} \to m$), i.e.,

we get $\dfrac{i}{\not{p} - m}$ for external on-shell

fermion ... $\Rightarrow$ mass of fermion

"remains" m (classical value)

[but could have chosen finite part of CT differently]