(superficial) Degree of divergence (D) of diagram / amplitude

– quickly / generally / systematically determine if diagram is UV-divergent or not (again, from $k \to \infty$ in $\int d^4 k_{loop}$) and (if yes) then to what degree (logarithmic, like vertex correction, or power ...)

[There are also IR divergences: $k_{loop} \to 0$ or $\underset{(final)}{p_{external}} \to 0$, i.e., soft emissions: see Phys 851 or QFT textbooks (briefly in HW 2.1)]

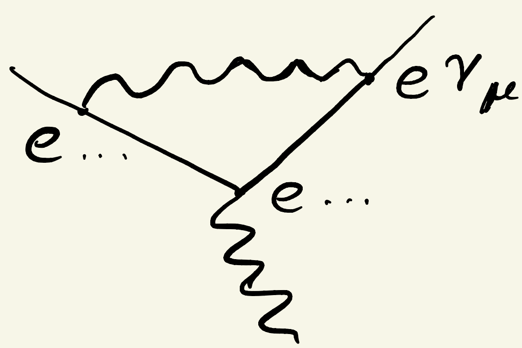
– following discussion somewhat abstract (mathematical), so to get intuition, just choose example of vertex correction

– Notation first: ℓ loops; $n_{f,b}$ number of internal fermion/boson lines (since $\sim \frac{1}{k}$ for fermion, $\frac{1}{k^2}$ for boson);
v_i number of times i^{th}-type vertex appears; d_i number of derivatives in i^{th}-type vertex

– QED: $i = 1$, $d_1 = 0$ (only $\bar{\Psi} A_\mu \Psi e$)

... but $\boxed{\text{scalar}}$ QED: $i = 1, 2$; $d_1 = 1$; $d_2 = 0$
[$\partial_\mu \phi A^\mu \phi$ & $(\phi A_\mu)^2$: see problem 9.1 of PS]

– 1-loop vertex correction: $\ell = 1$;
$n_b = 1$; $n_f = 2$; $v_1 = 3$



$\Rightarrow$ (superficial) degree of divergence (D) from counting powers of loop momenta in integral (neglect external momenta

— as appropriate — in UV-limit/∞ loop momenta) ...(1)

$$D = 4\,l - 2\,n_b - 1\,n_f + \sum_i v_i\,d_i$$

from $\left[\int d^4 k_{loop}\right]^l$ $\left[\frac{1}{k^2_{loop}}\right]^{n_b}$ $\left[\frac{1}{k^1_{loop}}\right]^{n_f}$ from ... $\left[\partial_\mu \sim k^1_{loop}\right]$

e.g., vertex correction: $D = 0$ (log-divergent)
[as shown earlier]

$D > 0$: power-divergent; $D < 0$: finite

— "Massage" above D into more convenient form, since it's difficult to quickly determine number of internal lines ($n_{b,f}$) for large number of loops (l): in any case, we'd like an all-loop order formula for D ...

... "expect" $n_{b,f}$, l to be related to number of external lines (more tractable):

Indeed (pure) "graph theory/math" (see exercise 6.6 of LP) gives

$$\sum_i v_i - n_b - n_f - 1 = -\ell \quad \ldots (2)$$

[sanity check: for 1-loop vertex correction, $v_1 = 3$; $n_b = 1$; $n_f = 2$; $\ell = 1$]

(More math)

– Also, let f_i denote number of fermionic fields involved in $i^{\underline{th}}$ type vertex, say, "coming-out" $\Rightarrow$

$v_i f_i$ fermionic lines from v_i of these vertices; $\sum_i v_i f_i$ in total...

...what's "fate" of these fermions: each either becomes external line (whose other end is "dangling") or internal line, whose other end "soaks-up" another fermion emerging from vertex

– So, we have ("matching" of fermionic lines from vertices with internal/external lines):

$$E_f + 2 n_f = \sum_i v_i f_i \quad \cdots (3)$$

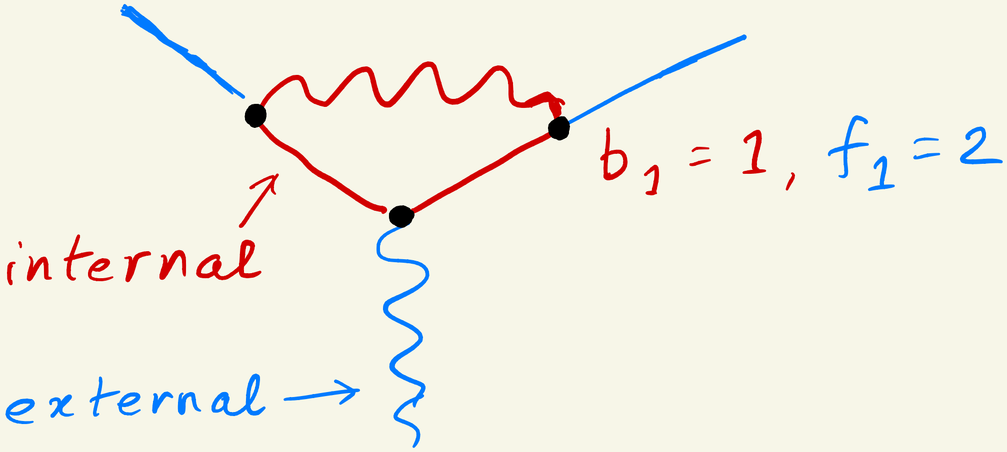
↑ number of external fermions

(Again, internal fermion line "uses-up" 1 fermion from vertex, while external needs 2 …)

e.g., 1-loop vertex correction: $v_1 = 3$; $E_b = 1$;

$n_b = 1$; $b_1 = 1$ & $E_f = 2$; $n_f = 2$; $f_1 = 2$

(only 1 type of vertex with 2 fermions, 1 boson appearing thrice)



Similarly for bosons,

$$E_b + 2n_b = \sum v_i b_i \quad \cdots (4)$$

number of external boson lines

number of bosonic fields at i^{th}-type vertex

Combining (1) thru' (4) gives (eliminating ℓ, $n_{b,f}$ in favor of $E_{b,f}$)

$$\boxed{D = 4 - E_b - \frac{3}{2}E_f - \sum_i v_i \delta_i},$$

where $\boxed{\delta_i = 4 - \left(d_i + b_i + \frac{3}{2}f_i\right)}$

Back to physics! what's meaning of δ_i?

Recall (mass) dimension of bosonic or fermionic field is 1 or $3/2$; each derivative "contributes" 1 ...

... so, b_i/f_i bosonic/fermionic fields & d_i derivatives at i^{th}-type vertex add up to dimension $\left(\frac{3}{2} f_i + b_i + d_i\right)$

$\boxed{+}$ dimension of coupling constant

= net mass dimension of term in Lagrangian $= [\mathcal{L}] = 4$

$$\Rightarrow \delta_i = 4 - \left(d_i + b_i + \frac{3}{2} f_i\right)$$

is (mass) dimension of i^{th}-type coupling constant

2 cases (by power-counting)

(1). Renormalizable theories: all $\boxed{\delta_i \geq 0}$

"large" $E_{b,f}$ gives $D < 0$ (finite), no matter loop-level (i.e., v_i: number of vertices)

$\Rightarrow$ $D \geq 0$ (divergence) only for finite/small number of cases (small $E_{b,f}$)

$\Rightarrow$ In short, tame these by adding CT's (to cancel divergences), but then can't predict these amplitudes (get from data)

- $\boxed{\text{Other}}$ amplitudes (convergent), that too a large number of them ("most/infinite": large $E_{b,f}$) can be predicted

- In more detail (still schematic), say loop contribution to amplitude

$$\mathcal{A}_{\text{loop}} \simeq + \log \Lambda_{UV} + x$$

(fixed, up to "mild" dependence on Λ_{UV}, e.g., $\sim \frac{1}{\Lambda_{UV}}$)

So, add CT with coefficient, A_{CT}

$= - \log \Lambda_{UV}$ (no choice here, since must cancel $+ \log \Lambda_{UV}$ from loop)

$+ y$ (arbitrary)

2 sub-cases (same end result)

(i) There exists tree/classical-level term in $\mathcal{L}$ (coefficient z) of same form as CT $\Rightarrow$ total amplitude, $A = z + y + x$

z: tree (free parameter)

y: arbitrary

x: fixed (loop: finite as $\Lambda_{UV} \to \infty$)

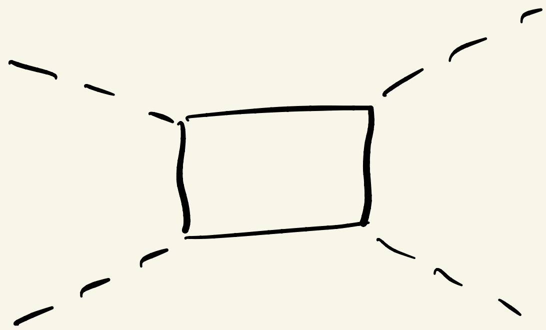
$\Rightarrow$ this amplitude is a free parameter (to be gotten from data)

(ii) There is no tree/classical level term of CT form ($z = 0$)

$\Rightarrow$ amplitude $= x + y$, still can't predict ...

e.g., $\boxed{\text{Yukawa}}$ theory: $\mathcal{L}_{int} = h\bar{\Psi}\psi\phi$,
but no ϕ^4 at tree/classical level

$\Rightarrow$ get ϕ^4 at loop-level (log-divergent):



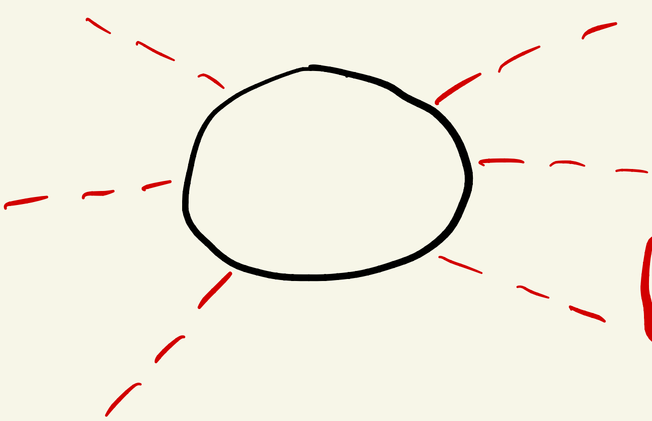
$D = 0$ due to
δ_1 (of h) $= 0$;
$E_b = 4$; $E_f = 0$

(At 1-loop: $\int d^4k \left(\frac{1}{k}\right)^4 \sim \log \Lambda_{UV} \ldots$)

[D independent of v_i - given $\delta_i = 0$, thus loop-number]

So, can't predict $\phi\phi \to \phi\phi$ ...

... but loop-induced ϕ^6 is convergent:



$D = -2$ due to
$\delta_1 = 0$; $E_b = 6$; $E_f = 0$

[At 1-loop, $\int d^4k \left(\frac{1}{k}\right)^6$
$\Rightarrow$ finite]

$\Rightarrow$ $\phi\phi \to \phi\phi\phi\phi$ or $\phi\phi\phi \to \phi\phi\phi$

... can be predicted

(2) (Power-counting) Non-renormalizable

if (any) $\boxed{\delta_i < 0}$ (coupling constant of negative mass dimension) $\Rightarrow$

For every amplitude (no matter what's given E_b, f), D "can be made" < 0 by choosing large enough v_i (number of vertices, i.e., higher loops) $\Rightarrow$ need to add CT for every amplitude, introducing a free parameter (y), even if no corresponding tree-level term ...

... so, even though divergences can be "cancelled", we lose all predictivity

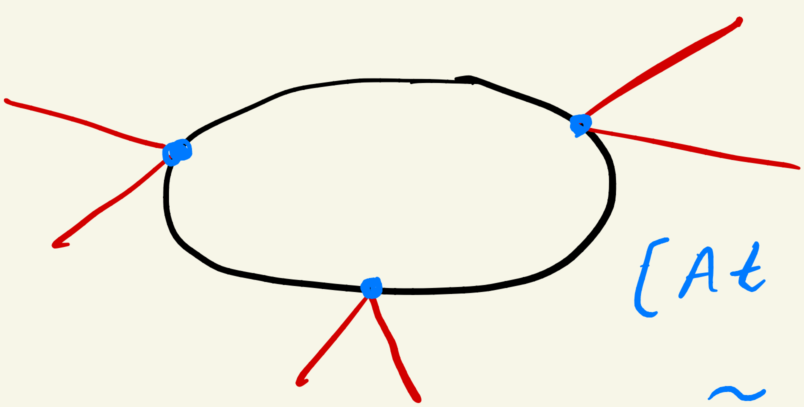
[divergences are unrelated /different

D's in general, so 1 CT is not enough, i.e., can't reduce number of free parameters to recover prediction...]

e.g., Fermi theory for weak interactions (return to it for motivating SSB / Higgs mechanism):

$\mathcal{L}_{int} \sim G_{F(ermi)} \psi^4 \Rightarrow \delta_1 (\text{of } G_F) = -2$

... so, ψ^6 diverges at 1-loop:



$D = +1$ due to $E_f = 6$; $E_b = 0$; $v_1 = 3$; $\delta_1 = -2$

[At 1-loop, $\int d^4k \left(\frac{1}{k}\right)^3 \sim \Lambda_{UV}^1$]

[D does depend on v_i - given $\delta_i \neq 0$, thus loop-number]

hence can't predict $\psi^3 \to \psi^3$, even if G_F fixed from data