Ward-Takhashi (WT) identity

— WT identity is diagrammatic (amplitude-level) expression/version of current conservation, in turn, following from gauge invariance:

$$\mathcal{L} = \bar{\Psi}(i\cancel{D} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $D_\mu = \partial_\mu + ieQA_\mu$ $(\cancel{D} = \gamma^\mu D_\mu)$

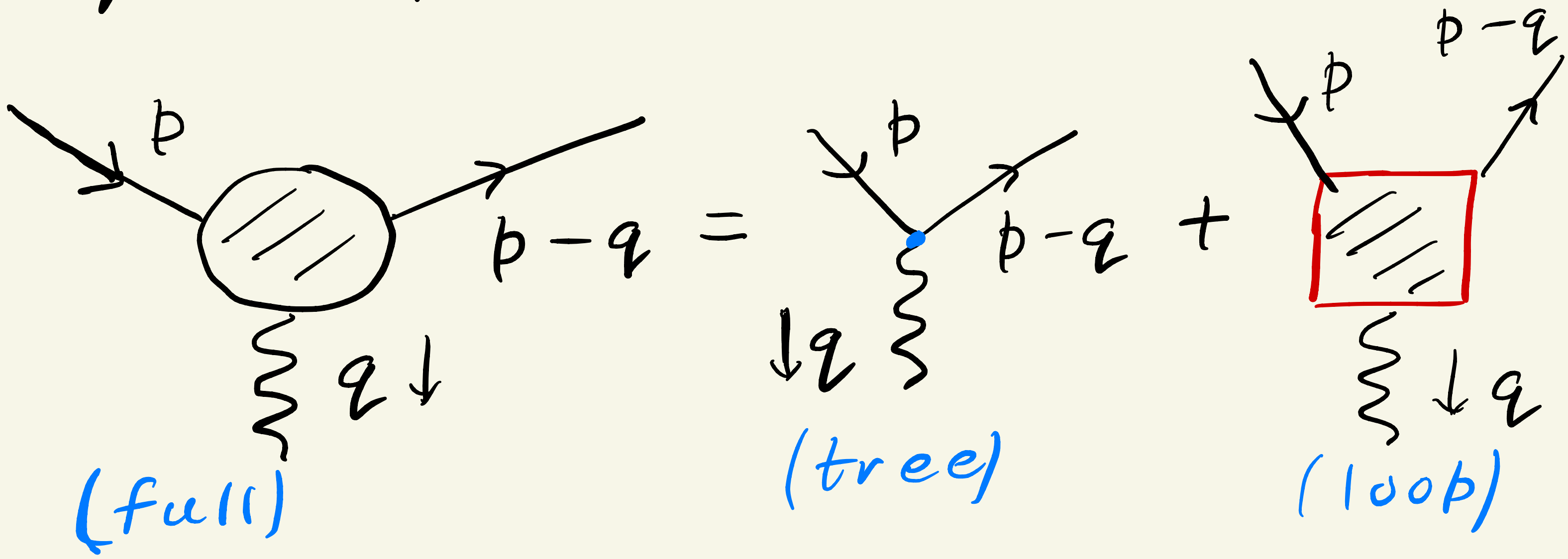
is invariant under

$$A_\mu \longrightarrow A_\mu + \partial_\mu \theta \quad \text{and} \quad \psi \rightarrow e^{-ieQ\theta(x)}\psi$$

[coupling to photon (A_μ) from $\boxed{\partial_\mu \longrightarrow \partial_\mu + ieQA_\mu}$]

— Notation: $\boxed{-ie\Gamma_\mu(p, p-q)}$ is amplitude for full (tree + loop) vertex/3-point function (without counting external legs/propagators)

$$\Gamma_\mu = \Gamma_\mu^{(0)}(\text{tree}) + \Gamma_\mu^{(\text{loop})}$$



(tree-level) $\Gamma_\mu^{(0)} = Q\,\gamma_\mu$

(Feynman rule is $-ieQ\gamma_\mu$: $Q = -1$ for electron ; $e > 0$) $\Rightarrow$

$$\underbrace{q^\mu}_{\text{photon momentum}} \Gamma_\mu^{(0)}(p, p-q) = Q\,\not{q}$$

$$= Q\left[(\not{p} - m) - (\not{p} - \not{q} - m)\right]$$

$$=$$

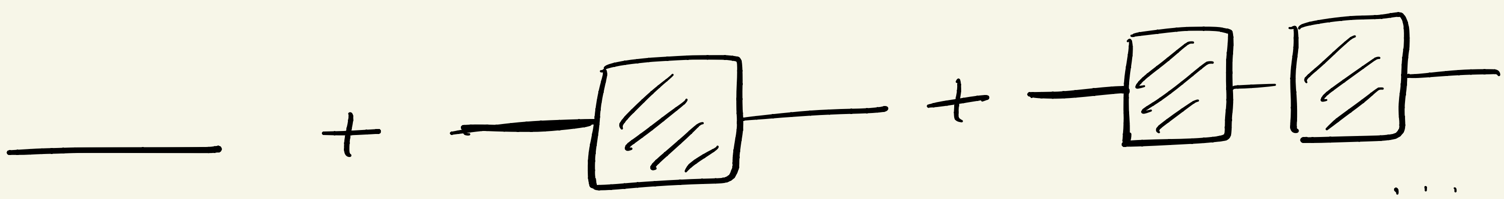
$S_F^{(0)} = \dfrac{1}{\not{p} - m} = \dfrac{\not{p} + m}{p^2 - m^2}$ is fermion propagator (Feynman rule is iS_F)

— WT identity : above valid at all loops ["remove" (0) superscript]

$$\boxed{q^\mu \Gamma_\mu(p, p-q) = Q\left[S_F^{-1}(p) - S_F^{-1}(p-q)\right]}$$

- Meaning of each side : Γ_μ as above
- $\boxed{i S_F}$ is propagator at all orders

$$= \underbrace{\frac{i}{\not p - m}}_{\text{tree}} + \frac{i}{\not p - m}\underbrace{\left[-i\Sigma(p)\right]}_{\text{fermion is 1PI self-energy}}\frac{i}{\not p - m} + \cdots$$



$\boxed{\text{1PI}}$ (one particle irreducible) cannot be split into 2 by cutting just 1 line

no tree

- Again, $\boxed{-i\Sigma(p)}$ (all loops) is Feynman amplitude for 2-point fermion function (not counting external line/ propagators)

- will show later : $\Sigma(p) \sim \not{p} + \mathbb{1}$ (as far as Dirac-space structure is concerned : coefficients not shown)

- So, $\Sigma(p)\ (\sim \not{p} + \mathbb{1})$

$$\frac{1}{\not{p} - m} = \frac{\not{p} + m}{p^2 - m^2}$$ $\Rightarrow$ sum geometric series (see CL sec 2.1 or PS sec. 7.1) to give

$$i S_F\ (\text{tree} + \text{all loops}) = \frac{i}{\left[\not{p} - m - \Sigma(p)\right]}$$

$\Rightarrow$ WT identity is really

$$q^\mu \left[\Gamma_\mu^{(0)} + \Gamma_\mu^{(loop)}\right] = Q\left\{\left[\not{p} - m - \Sigma(p)\right] - \left[\not{p} - \not{q} - m - \Sigma(p-q)\right]\right\}$$

all orders propagators

$$= \underbrace{Q \not{q}} - Q\left[\Sigma(p) - \Sigma(p-q)\right]$$

$$\Rightarrow \boxed{q^{\mu}\,\Gamma_{\mu}^{(\mathrm{loop})} = -Q\left[\overbrace{\Sigma(p) - \Sigma(p-q)}^{\text{loop}}\right]}$$

WT identity relates loop amplitudes (vertex correction & fermion self-energy): both being matrices in Dirac space

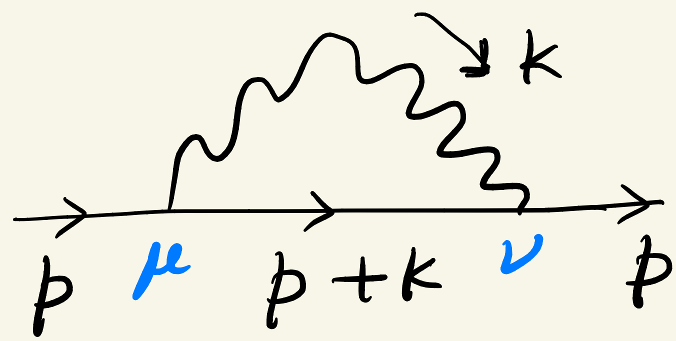
Physical intuition/implications: not so clear from above form, will show later leads to

(a) photon remaining massless even at loop-level and

(b) electron to proton electric charge ratio = 1 at loop-level, even though electron & proton have different other interactions (e.g. strong force)

$\boxed{\text{Proof}}$ at $\boxed{\text{one-loop}}$ with only QED interaction (without actually computing loops, which are actually divergent!)

$-i\Sigma^{(1)}(p)$ from

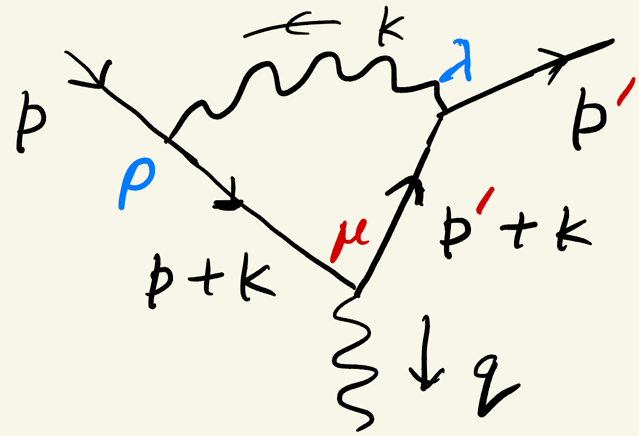


(matrix in Dirac space)

$$= (-ieQ)^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{i}{(\not{p} + \not{k} - m)} \gamma_\nu \; i D^{\mu\nu}(k)$$

photon propagator (depends on gauge fixing parameter ξ)

$-ie\,\Gamma_\mu^{(1)}$ from



$p' = p - q$

$$= (-ieQ)^3 \times \int \frac{d^4k}{(2\pi)^4}\, i\,D^{\lambda\rho}(k) \times$$

$$\gamma_\lambda \frac{i}{(\not p' + \not k - m)} \gamma_\mu \frac{i}{(\not p + \not k - m)} \gamma_\rho$$

so that $q^\mu \, \Gamma_\mu^{(1)}$ (entering WT identity)

$$= (-ieQ)^2 Q \int \frac{d^4k}{(2\pi)^4}\, i\,D^{\lambda\rho}(k) \times$$

$$\gamma_\lambda \frac{i}{(\not p' + \not k - m)} \underbrace{\left[(\not p + \not k - m) - (\not p' + \not k - m)\right]}_{q^\mu \gamma_\mu} \times$$

$$\frac{i}{(\not p + \not k - m)} \gamma_\rho$$

$$= (-ieQ)^2 Q \int \frac{d^4k}{(2\pi)^4}\, i\,D^{\lambda\rho}(k) \times$$

$$\gamma_\lambda \; i\left[\frac{i}{(\not{p}' + \not{k} - m)} - \frac{i}{(\not{p} + \not{k} - m)}\right]\gamma_\rho$$

$$\Rightarrow q^\mu \Gamma^{(1)}_\mu(p, p') = Q\left\{-\Sigma^{(1)}(p) - \left[-\Sigma^{(1)}(p')\right]\right\}$$

... which is (loop form of/ WT identity

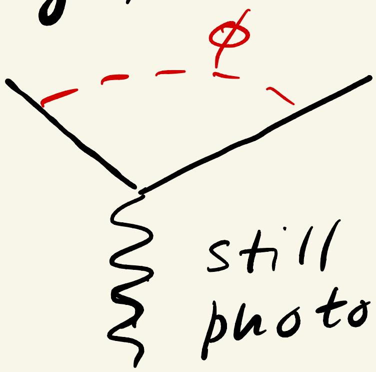
Some comments :

(1). valid for all ξ (gauge fixing)

(2). more general proof from gauge invariance / current conservation :

(a). all loops in QED only

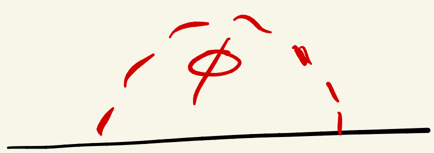
(b). other gauge-invariant interactions, e.g., add $h\bar{\Psi}\Psi\phi$ (Yukawa coupling):



still photon

$\Gamma^{(1)}_\mu$ but ϕ loop

vs.

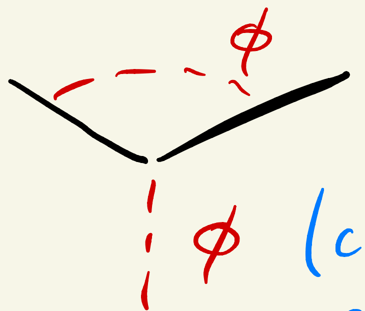


$\Sigma^{(1)}$ from ϕ loop

[HW 1.1 : similar manipulation to above ...]

(3). Again, gauge invariance (coupling to photon from $\partial_\mu \to \partial_\mu + ieQA_\mu$) crucial ...

... so, no analogous identity for (pure) Yukawa theorgy, i.e., coupling to ϕ: cannot relate (at least at all orders)



(cf. photon earlier)

to