$\boxed{\text{Spontaneous Symmetry Breaking (SSB)}}$:

case of $\boxed{\text{global}}$ (vs. gauged/local later) symmetry, with 2 sub-cases: $\boxed{\text{discrete}}$ and $\boxed{\text{continuous}}$

Outline for $\boxed{\text{global}}$ SSB:

- simplest model is $\boxed{\text{discrete}}$ symmetry, illustrates the following general features: Lagrangian has symmetry, but not realized in physically, i.e., vacuum/ground state does not respect it.
- relatedly vacuum state is not unique (there's degeneracy): non-vanishing quantity (order parameter) characterizes necessary choice

... however, symmetry is still "hidden" interactions display a "remnant" effect of symmetry (again, this is not apparent/manifest/obvious)

— Onto, $\boxed{\text{continuous}}$ (global) symmetry: (new feature) massless field/particle called Nambu-Goldstone boson (NGB), which is derivatively-coupled

$\boxed{\text{Discrete global SSB model}}$ [only real scalar field (ϕ)]

$$\mathcal{L} = \underbrace{\frac{1}{2}\left(\partial^\mu\phi\, \partial_\mu\phi\right)}_{\text{K(inetic): derivative part}} - \underbrace{\left(\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4\right)}_{\text{V (potential): non-derivative}}$$

[K, V in analogy to classical mechanics, even though here it's not that force $= -\partial V/\partial x$ etc.!]

— Discrete (Z_2) symmetry: $\phi \to -\phi$ (due to no ϕ^1 or ϕ^3 term in $\mathcal{L}$)

(no infinitesimal version)

(More physical meaning — in terms of spin-0/scalar particles upon quantization)

— Above $\mathcal{L}$ gives Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\left[\pi^2 + (\bar{\nabla}\phi)^2\right] + V(\phi), \text{ where}$$

$\pi = \dot{\phi}\ (\equiv \partial\phi/\partial t)$ is canonical momentum conjugate to ϕ

$\boxed{\text{Classical}}$- level analysis first [think of ϕ as background/classical field (like $\bar{E}, \bar{B}$)]: what $\phi(\bar{x}, t)$ will minimize above $\mathcal{H}$/energy (i.e., ground state)?

Since $(\bar{\nabla}\phi)^2, (\pi = \dot{\phi})^2 \geq 0$, it's clear that we should choose ϕ = constant ($\bar{x}, t$ independent): Poincare-invariant

And, determine this ϕ by minimizing $V(\phi)$:

$d V/d\phi = \phi(\mu^2 + \lambda\phi^2) = 0$ and

$d^2V/d\phi^2 = \mu^2 + 3\lambda\phi^2 > 0$

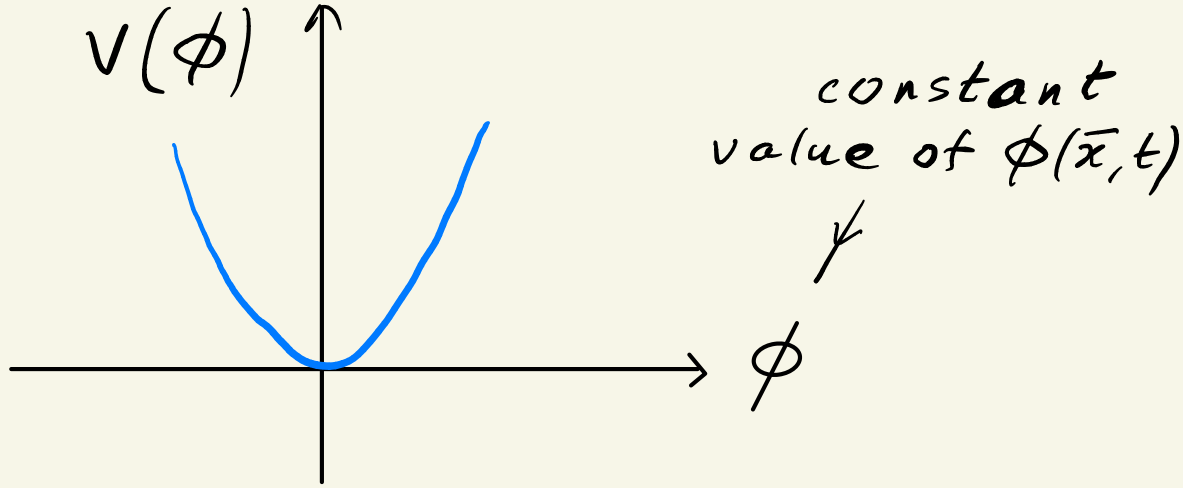
$\Rightarrow$ 2 cases, depending on sign of μ^2 (mass term): (i) $\mu^2 > 0$ & (ii) $\mu^2 < 0$

What about sign of λ (quartic "coupling")?

For large ϕ, $\lambda\phi^4$ dominates $\Rightarrow$ for energy to be bounded from below, choose

$\lambda > 0$ (λ is real for $\mathcal{H}$ to be Hermitian)

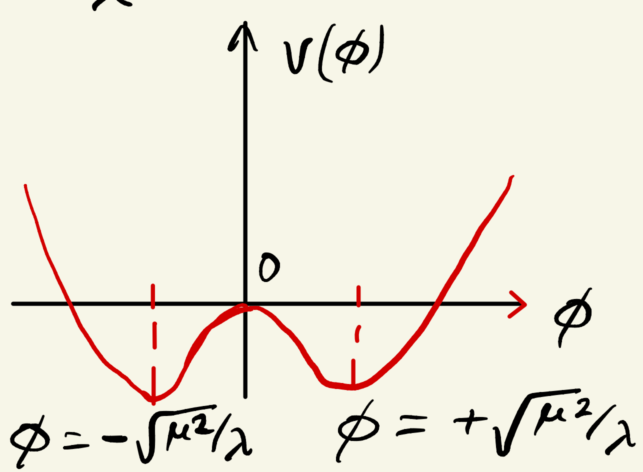
(i). $\mu^2 > 0 \Rightarrow \phi = 0$ at minimum (no other extremum)



Symmetry is unbroken in ground state, e.g., ferromagnet above Curie temperature: spins randomly oriented (no preferred direction $\Rightarrow$ 3d rotational symmetry)

(ii) $\mu^2 < 0 \Rightarrow \phi = 0$ is maximum;

$\phi = \pm\sqrt{-\mu^2/\lambda}$ is minimum:



$\Rightarrow$ Ground state is either $\phi = +\sqrt{-\mu^2/\lambda}$ or $\phi = -\sqrt{-\mu^2/\lambda}$ (classically, cannot have a superposition) $\Rightarrow$ $\phi \to -\phi$ symmetry is broken

– vacuum not unique: non-zero order parameter is $\pm\sqrt{-\mu^2/\lambda}$, e.g. ferrowmagnet below Curie temperature: spins aligned in same direction $\Rightarrow$ 3d rotational symmetry broken in ground state by this selection of direction, but which could be arbitrary (with same energy): degeneracy (infinite, so more like continuous symmetry)

On to quantizing above classical field:

ϕ classical value above [0 for $\mu^2 > 0$ & $\pm\sqrt{-\mu^2/\lambda}$ for $\mu^2 < 0$] is vacuum expectation value (VEV) of quantum field/operator: $\langle 0 | \phi | 0 \rangle$ or just $\langle \phi \rangle$

$\underbrace{\langle 0 |}_{\text{no particles in vacuum}}$; $\phi \to$ quantum field

(will give more physical meaning to SSB)

(i). $\mu^2 > 0$ case (usual, i.e., as in very introductory QFT): expand ϕ as

$$\phi(x_\mu) \sim \int d^3p \left(\underbrace{a}_{\substack{\text{annihilation}\\ \text{operator}\\ \text{(quantization)}}} e^{-ip\cdot x} + \underbrace{a^\dagger}_{\substack{\text{creation}\\ \text{operator}}} e^{+ipx} \right)$$

$a^\dagger$ "annihilates"

(consistent with $\langle 0 | \phi | 0 \rangle = 0$)

a annihilates...

$\Rightarrow \mu^2 (>0)$ becomes (physical) mass of spin-0/scalar particles (quanta or excitations of ϕ): energy in $\bar{p}$ mode $= n \sqrt{\mu^2 + |\bar{p}|^2} \ldots$

→ (positive) integer: number of particles

... interacting via $\lambda\phi^4$ ...

Since $\phi \to -\phi$ is symmetry of vacuum, we see that a process involving odd number of ϕ-particles is not allowed

(ii) $\mu^2 < 0$ (SSB): if ϕ expanded as above, then $\langle\phi\rangle = 0$ vs. $\pm\sqrt{-\mu^2/\lambda}$ at minimum ...

... or, $(\text{mass})^2$ of particles, $\mu^2 < 0$ (from energy in $\bar{p}$ mode $= n\sqrt{|\bar{p}|^2 - |\mu^2|}$)

— this inconsistency is expected since we expanded ϕ around "wrong" vacuum ($\langle\phi\rangle = 0$, which is unstable)

— So, expand ϕ around minimum: $\pm\sqrt{-\mu^2/\lambda} \equiv \pm v$: pick either vacuum [aside: at quantum-level, cf. classical earlier, can we choose vacuum to be a superposition of $\langle\phi\rangle = \pm v$? For example, a 50-50 admixture could preserve symmetry!

No: see Weinberg section 19.1 (p. 164–165) posted online.]

– Shift field: $\boxed{\phi(x) = v + \eta(x)}$

so that $\langle \eta \rangle = 0$, thus expand it as usual:

$$\eta(x) \sim \int d^3p \left(a\, e^{-ip\cdot x} + a^\dagger e^{+ip\cdot x} \right)$$

– Rewrite $\mathcal{L}$ in terms of η (using $\mu^2 = -\lambda v^2$, dropping constant term in $\mathcal{L}$), i.e., fluctuations around VEV:

$$\mathcal{L}_{SSB} = \tfrac{1}{2}(\partial^\mu \eta)(\partial_\mu \eta) - \left(+ \lambda v^2 \eta^2 + \lambda v \eta^3 + \frac{\lambda}{4}\eta^4 \right)$$

Two features:

(a) $m^2_\eta \equiv 2\lambda v^2 > 0$ (vs. when ϕ expanded around $\langle\phi\rangle = 0$) : (only) particles, η, are massive (cf. continuous symmetry case next)

(b) Due to η^3, processes with odd number of η-particles are allowed, i.e., $\eta \rightarrow -\eta$ symmetry is broken/not realized (at least not manifest/apparent) physically (in interactions) ...

(c) Remarkably, symmetry still partly survives (i.e., is "hidden") as follows. Compare above $\mathcal{L}$ to the case of explicit breaking, rather, no symmetry at all to begin with:

$$\mathcal{L}_{explicit} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + M^3 \eta - \mu^2 \eta^2 + \mu' \eta^3 + \lambda/4\, \eta^4$$

which looks somewhat similar to $\mathcal{L}$ after SSB, except for crucial difference that SSB case obtained from general $\mathcal{L}$ by (1). setting $M^3 = 0$ and (2). μ' (i.e., coefficient of η^3: cubic coupling) is related to η^2 (mass term) and η^4 (quartic coupling): again 4 (independent) parameters (M', μ', μ^2 & λ) in $\mathcal{L}$ explicit vs. only 2 (λ and μ^2 or m_η^2 & λ) in $\mathcal{L}_{SSB}$ (so, "remnant" symmetry in the form of relation between cubic coupling and mass, quartic...)

— Keep in mind above as general theme of SSB, i.e., "gentle/controlled" breaking of symmetry vs. explicit, more general : e.g., ultimate goal of SSB is to generate gauge boson mass, where its "survival" of renormalizability which is crucial (cf. explicit breaking of gauge symmetry by bare mass term does not retain this feature