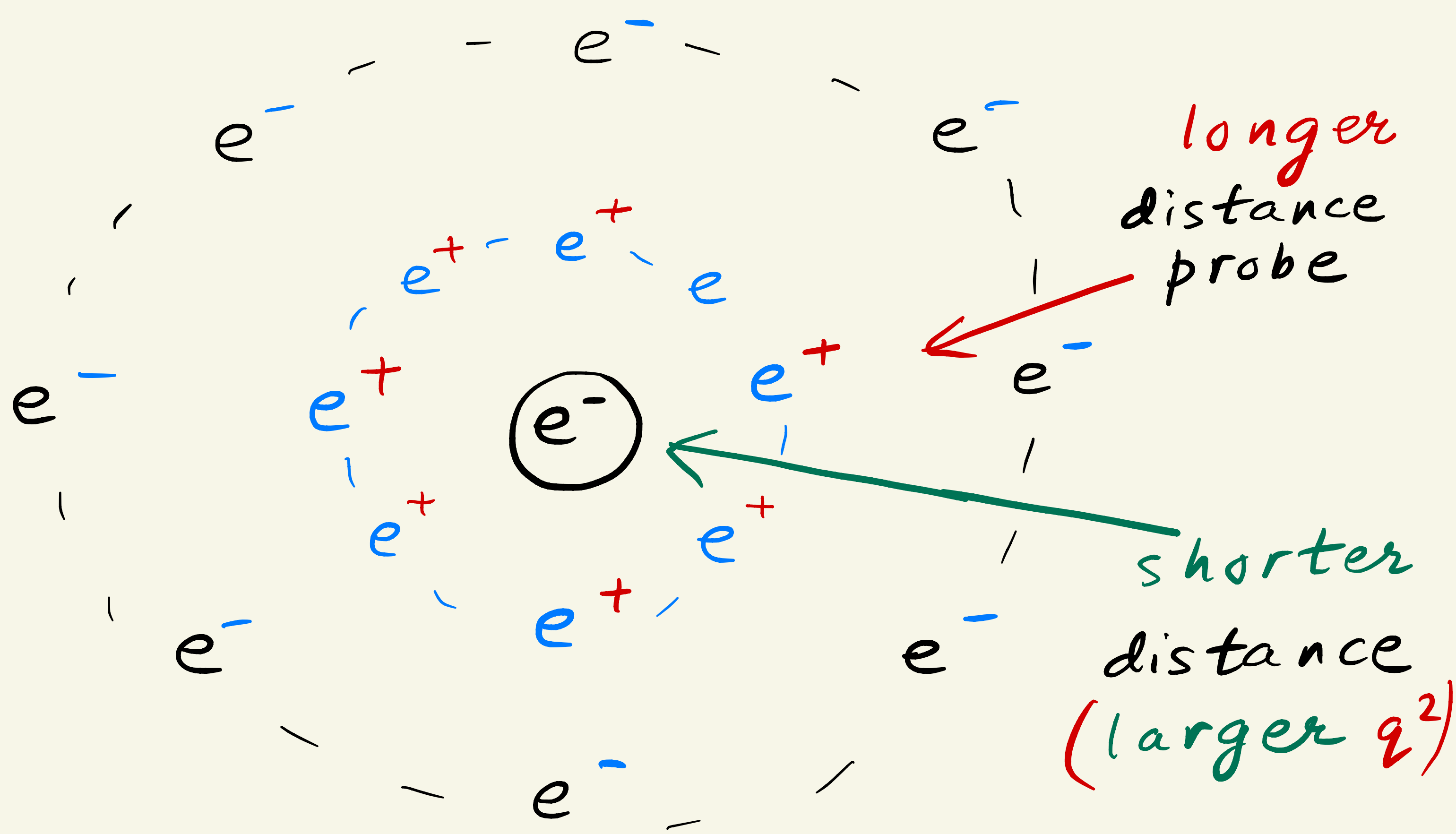
"Physical" consequence of renormalization

$\boxed{\text{Running of effective QED coupling}}$ ("from" $\Pi_{\mu\nu}$)

Intuitive/heuristic understanding/expectation
(why $\Pi_{\mu\nu}$ is called "vacuum polarization")

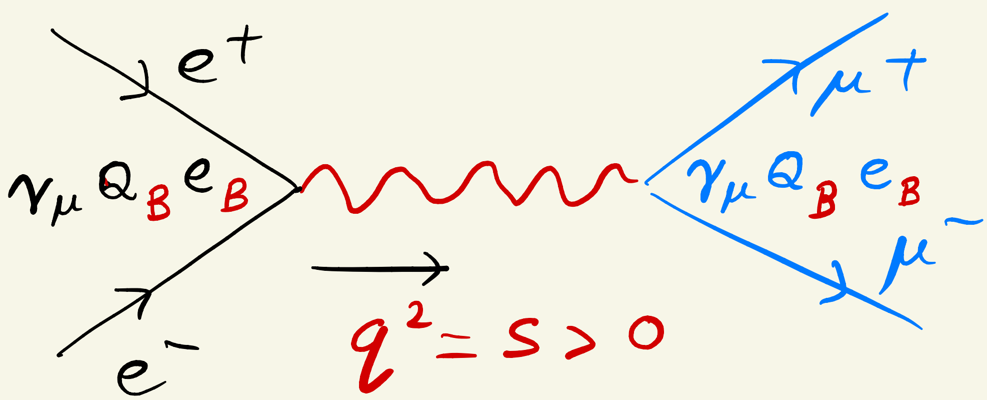
- (isolated) electron "polarizes" surrounding vacuum: virtual e^- further away from real electron (virtual e^+ closer)



$\Rightarrow$ at longer distances (smaller q^2), (bare) electron charge screened/reduced by virtual e^+ cloud ... vs. at shorter distances (larger q^2), cut through e^+ cloud to probe full charge ...

$\Rightarrow$ expect charge/coupling constant to become larger for higher q^2 (IR-free) ... will show it is explicitly due to $\Pi_{\mu\nu}$ (hence vacuum polarization)

– Use $\boxed{e^+e^- \to \mu^+\mu^-}$ [more formal treatment (renormalization group equation etc.) in Phys 851...]



$\sqrt{s}$ (center-of-mass energy) $\gtrsim$

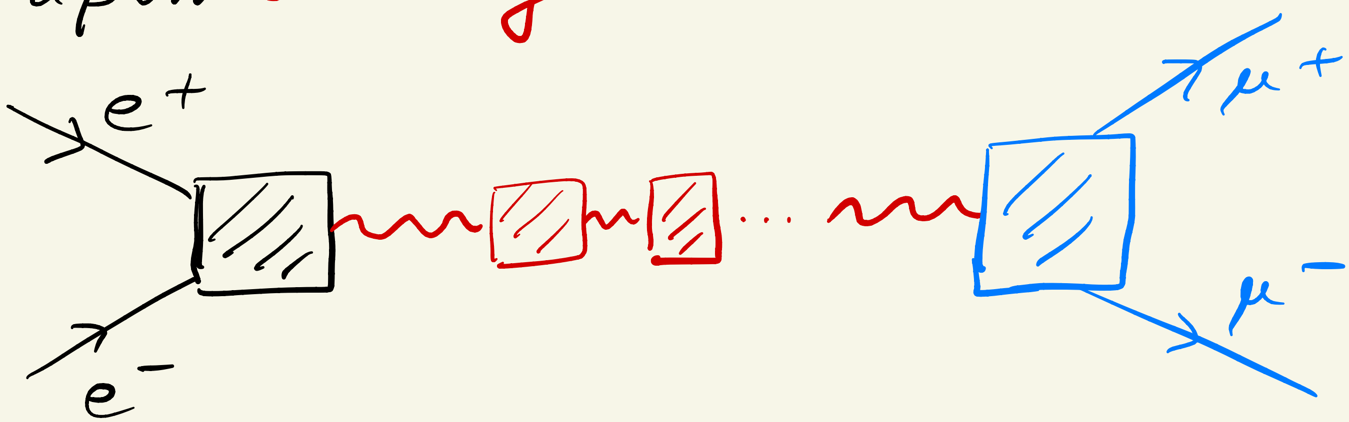
– Start with $\mathcal{L}_B$ (with e_B, ψ_B, A^μ_B), i.e., conventional renormalization

– amplitude (including spinors for external, -shell, fermions):

at tree-level $\sim \left[(\bar{v}_B)_e \gamma^\mu e_B (Q_B)_e (u_B)_e \right] \times$

$$\times \underbrace{1/q^2}_{\text{photon propagator}} \left[(\bar{u}_B)_\mu \gamma^\mu e_B (Q_B)_\mu (v_B)_\mu \right]$$

$\longrightarrow$ at loop-level (not complete: see in a bit!), i.e., upon adding



$\sim$ (being schematic about Dirac structure)

$\left[(\bar{v}_B)_e \ldots (v_B)_\mu \right]$ ("product" of spinors) $\times$

$$\left[(Q_B)_e \gamma^\mu + (\Gamma_\mu^{\text{loop}})_e \right] \left[(Q_B)_\mu \gamma_\mu + (\Gamma^{\mu \text{ loop}})_\mu \right] \times$$

$$(e_B)^2 \frac{1}{q^2}\left[\frac{1}{1-\pi(q^2)}\right]$$ resumming $\pi_{\mu\nu}$ "insertions" on photon line

- Focus on (cancellation of) divergences, in bare parameters and loop contributions:

$$e_B^2 \Big/ \underbrace{\left[1-\pi(q^2)\right]}_{\text{divergent}} = \frac{e^2 \text{(observed/finite)}}{Z_3 \text{(divergent)}} \Big/ \left[1 - \left(-\frac{\alpha}{3\pi}\frac{1}{\varepsilon'} + \text{finite, } q^2\text{-dependent}\right)\right]$$

q^2-dependent: drop for now

with $Z_3 = \left[1 - \frac{\alpha}{3\pi}\frac{1}{\varepsilon'} + \text{finite, } q^2\text{-independent}\right]$

$\simeq e^2$, dropping $\mathcal{O}(\alpha^2)$ in product

[So, divergences cancel between Z_3 & π, just like using CT's, i.e, BPH renormalization]

- Similarly, $\left[(Q_B)_e \gamma_\mu + (\Gamma_\mu^{\text{loop}})_e\right]$

$$\simeq \underbrace{Q_e}_{\text{observed}} {}^{(Z_1)_e}\!\big/_{(Z_2)_e}\, \gamma_\mu - \underbrace{Q_e \gamma_\mu (Z_1 - 1)}_{\text{drop finite parts of } \Gamma_\mu^{\text{loop}}},$$

using divergence in Γ_μ^{loop} cancelled by Z_1 CT (for all q^2, i.e., not just at $q^2 \to 0$, where CT was chosen)

- Next, use $Z_1 = Z_2$ (from WT identity) in both terms above to get

$$= Q_e (2 - Z_2) \gamma_\mu , \text{ with } \boxed{Z_2 = 1 + O(\alpha)}$$

$$= Q_e [1 - O(\alpha)] \simeq \frac{Q_e}{[1 + O(\alpha)]} = \boxed{\frac{Q_e}{Z_2}}$$

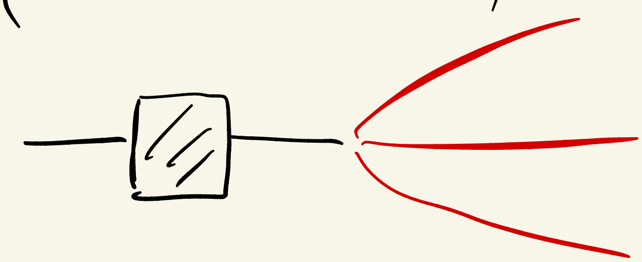
- So, as far as divergences are concerned,

$$\text{full amplitude} \sim \underbrace{\frac{e^2}{q^2} Q_e Q_\mu}_{\text{finite}}$$

$$\times \frac{\left[(\bar{v}_B)_e \gamma_\mu (u_B)_e\right]}{(Z_2)_e} \frac{\left[(\bar{u}_B)_\mu \gamma^\mu (v_B)_\mu\right]}{(Z_2)_\mu} \ldots$$

... so, still left with divergences in $(Z_2)_{e,\mu}$?!

... well, not quite done yet (as "warned" at start): what about dressing of external (on-shell) fermion lines? But, didn't we say (in BPH renormalization) that can be neglected? Yes, but that was when we started with $\mathcal{L}_{classical}$, vs. $\mathcal{L}_B$ here, i.e., for bare external fermion lines, there is (non-trivial) dressing:



rest of diagram

i.e., just like for field ($\psi_B = \psi \sqrt{Z_2}$), we have $\boxed{u_B = \underbrace{u}_{\text{"observed"}} \sqrt{Z_2}}$, where u's are properly normalized (u_B are not):

$$\boxed{\Sigma_{spins}\, u \bar{u} = \not{p} - m \ (\text{observed})}$$

– So, indeed after dressing bare amplitude by loops as above, we get finite result:

$$\sim \left[(\bar{v})_e \gamma^\mu (u)_e \right] \left[(\bar{u})_\mu \gamma_\mu (v)_\mu \right] Q_e \, Q_\mu \, \frac{e^2}{q^2},$$

where $\sum_{\text{spins}} \bar{u} u = \not{p} - m$ etc. (m_e, m_μ, e, Q_e & Q_μ being observed / measured parameters)

Summary, after all of the song-and-dance, prescription seems to be "simply remove" B subscript from everywhere in tree amplitude: $e_B \to e$, $u_B \to u$, $Q_B \to Q$...

... but that's not complete story, once we look beyond divergences, i.e., there is a significant, finite, remnant effect from $\pi(q^2)$ which can be predicted / tested as follows.

– Recall that effects of $Z_{1,2}$ (including any large, finite) "cancel" each other, so we are left with $\pi(q^2)$ to consider:

indeed, based on above discussion, we can define an "effective" coupling constant, $e_{eff}(q^2)$ as

$$\frac{e_B^2}{q^2\left[1-\pi(q^2)\right]} \quad \text{(part of above amplitude)}$$

$$= e^2_{eff}(q^2)/q^2$$

in such a way that "prescription" to include loop effects into "bare" amplitude is (as outlined above) $\boxed{e_B \rightarrow e}$ (at level of $\boxed{\text{divergence}}$), where $e^2 = 4\pi\alpha_{QED}$ (with $\alpha_{QED} = \frac{1}{137}$ as measured at $q \rightarrow 0$) ... and at "next" level (i.e., $\boxed{\text{finite}}$, dominant effect) we also have $\boxed{e \rightarrow e_{eff}(q^2)}$ ("keeping" photon propagator $1/q^2$ "as is")

— So, we have

$$e^2_{eff}(q^2) = e^2_B/\left[1-\pi(q^2)\right] = \frac{e^2}{Z_3\left[1-\pi(q^2)\right]}$$

(again, e^2 here is as observed at very low energies, e.g., atomic systems)

— This is same as earlier, but it's just that we now keep track of large, finite effects in Z_3 and $\pi(q^2)$, that too from multiple fermions (labeled by "i", e.g., $i =$ electron, muon or tau)

$$= \frac{e^2}{\underbrace{\left[1 - \frac{\alpha}{3\pi} \sum_i Q_i^2 \left(\frac{1}{\varepsilon'} - \log \frac{m_i^2}{\mu^2}\right)\right]}_{Z_3} \underbrace{\left\{1 + \frac{2\alpha}{\pi} \sum_i Q_i^2 \left[\frac{1}{6\varepsilon'} - I_i(q^2)\right]\right\}}_{-\pi(q^2)}}$$

(no q-dependence)

where $I_i(q^2) = \int dx\, x(1-x) \log\left[\frac{m_i^2 - x(1-x) q^2}{\mu^2}\right]$

We see that (as expected and as already outlined earlier) divergences ($\propto 1/\varepsilon'$) cancel between Z_3 and $\pi(q^2)$ ... so does

μ-dependence, since we have

$$I_i(q^2) = -\log \mu^2 \int dx\, x(1-x) + \ldots (\text{no } \mu^2)$$

$$= -\frac{1}{6} \log \mu^2$$

Hence we get (as usual dropping even higher order in α)

$$e^2_{eff}(q^2) \simeq e^2 \left[1 + \frac{2\alpha}{\pi} \sum_i Q_i^2 I_i(q^2) \right]$$

so that "running" effect, i.e, difference in (effective coupling constant)^2 at two different energies (q^2 vs. q'^2), is given by

$$\boxed{e^2(q^2) - e^2(q'^2) = e^2 \frac{2\alpha}{\pi} \sum Q_i^2 \left[I_i(q^2) - I_i(q'^2) \right]}$$

drop "eff" for simplicity

(actually, it doesn't really matter at what energy we evaluate e^2, α on RHS above, since that "difference" would be even higher order in α)

— So, we get 3 cases, depending on m (let $q^2 \gg q'^2$ without loss of generality) vs. q^2, q'^2 (both > 0, since they are s, s': CoM energies here)

(i). $\boxed{q^2, q'^2 \ll m^2} \Rightarrow I(q^2) - I(q'^2) \simeq 0$

since both I's $\simeq \int dx\, x(1-x) \log m^2/\mu^2$

(ii). $\boxed{q'^2 \ll m^2 \ll q^2} \Rightarrow$ (skipping details)

$$I(q^2) - I(q'^2) = \frac{1}{6}\left[\log\left(-\frac{q^2}{m^2}\right) - 5/3\right] > 0$$

Note: $\log(-1) = i\pi$, due to fermion-

i: imaginary; π: 3.142

antifermion "inside" $\Pi_{\mu\nu}$ going on-shell for $q^2 (= s) \geqslant 4m^2$

(iii). $\boxed{q^2 \gg q'^2 \gg m^2}$ gives

$$\underbrace{I(q^2) - I(q'^2)}_{\equiv \Delta I} = 1/6 \log(q^2/q'^2) > 0$$

Suppose $m_\tau^2 \gg q^2 \gg m_\mu^2$ ($m_\tau = 1.8$ GeV vs. $m_\mu = 0.1$ GeV), whereas

$m_\mu^2 \gg q'^2 \gg m_e^2$ ($m_e = 0.0005$ GeV)

i.e., we are comparing effective coupling constant to be used for $e^+e^- \to \mu^+\mu^-$ (but at energies where $e^+e^- \to \tau^+\tau^-$ is kinematically forbidden) to that in $e^+e^- \to e^+e^-$ at energies such that $e^+e^- \to \mu^+\mu^-$ is not allowed.

So, we use case (i) for τ, i.e., $\Delta I \simeq 0$;

(ii). for μ, i.e., $\Delta I \equiv I(q^2) - I(q'^2)$

$$\simeq \frac{1}{6}\left[\log\left(q^2/m_\mu^2\right) - 5/3\right] > 0$$

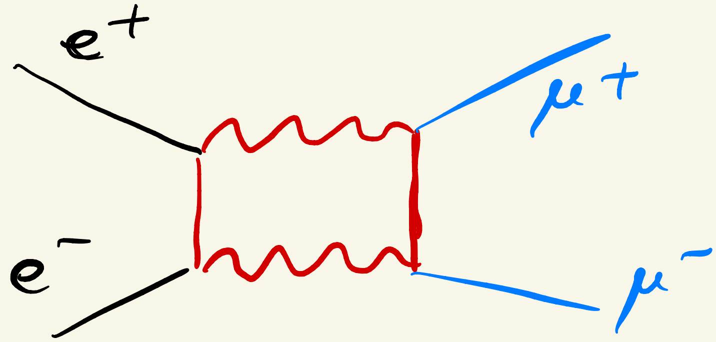
(iii). for electron, i.e., $\Delta I \simeq 1/6 \log\left(q^2/q'^2\right)$ (independent of m_e) > 0

Thus, $\boxed{\begin{array}{l} e^2 \text{ at } q^2 \gg m_\mu^2 \left(\text{but} \ll m_\tau^2\right) \\ > e^2 \text{ at } q'^2 \ll m_\mu^2 \left(\text{but} \gg m_e^2\right) \end{array}}$

(More in HW 2.2, being careful with factors of electric charges & color for quarks)

i.e., $\boxed{\text{IR - free running}}$

– What about finite loop diagram?



$(D = -2)$

– This is not enhanced by a large logarithm of ratio of mass/energy scales, cf. vacuum polarization effect above, where we could be evolving from $q^2 \sim m_e^2$ to $q^2 \sim M_W^2 \simeq (80 \text{ GeV})^2$ as in HW 2.2 so that $\log \frac{M_W}{m_e} \simeq 24$

– So, above $e_{eff}(q^2)$ keeps track only of dominant (log-enhanced) effect (for more precise treatment, see Phys 851 or PS