Putting it all together [step (3)]

(after regularization & counterterms):

$\boxed{\text{Renormalization}}$

(more complete/formal discussion in Phys 851)

– Recall we start with

$$\mathcal{L}_{\text{classical}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - eQ\bar{\psi} A\!\!\!/ \psi$$

... actually, "separating" coupling of photon to fermion into e (coupling constant) & Q (charge of fermion) has meaning only if there are > 1 fermions with different charges, e.g., electron ($Q = -1$) and quarks ($Q = -1/3, 2/3$) ... even then, we can "absorb" e into Q for 1 fermion (say electron) so that physically relevant (possibly predictable, including loops etc.) parameter is ratio of Q's

– With above $\mathcal{L}_{classical}$, loops have divergences: $\Pi_{\mu\nu}$; Σ and Γ_μ

– So, added counterterms (CT's) to cancel divergences (same form as $\mathcal{L}_{classical}$):

$$\mathcal{L}_{CT} = -\frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu} - (Z_1 - 1) e Q \bar{\psi} \not{A} \psi + (Z_2 - 1) \bar{\psi} \left[i \not{\partial} - (m - \delta m) \right] \psi$$

– Now, divergent part of CT's "fixed" by goal of canceling divergences from loops, but finite parts chosen such that

(a). full (classical/tree + loop + CT) photon propagator in on-shell limit $(q^2 \to 0)$ = same as classical (mass of photon remains zero)

(b). full fermion propagator =

classical (again) in on-shell limit

$\Rightarrow$ observed mass of fermion $= m$

(as in $\mathcal{L}$ classical); no "dressing"

of external on-shell fermion line

(c). For kinematical configuration

of both fermions being on-shell &

q (photon) $\rightarrow 0$ (really $q^2 \ll m^2$),

full vertex, $\Gamma_\mu (p^2 = p'^2 = m^2; q^2 \rightarrow 0)$

$\hookrightarrow$ tree + loop + CT

= classical, Q

Combining above 3 choices, i.e.,

vertex same as classical, but also

fermion / photon propagator = classical,

we see that

electric charge (even at loop-level)

$= eQ$ (= classical) ...

... but then there's "no effect" of loops (no renormalization of m or eQ) ?!

..., not quite (as follows)! Consider full Lagrangian (call it "bare"):

$$\mathcal{L}_{B(are)} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{CT}$$

(after all, both are terms in Lagrangian ... that too similar form...)

– We will see below that observed values (eQ, m: as mentioned above) $\neq$ those in $\mathcal{L}_B$ (even if same as $\mathcal{L}_{\text{classical}}$), i.e., there is renormalization...

– Begin with $\mathcal{L}_B$ as function of ψ, A (canonically normalized fields in $\mathcal{L}_{\text{classical}}$), but with coefficients of terms dependent on Z's (CT's) & eQ, m (observed values):

$$\mathcal{L}_B = -\frac{Z_3}{4} F_{\mu\nu}F^{\mu\nu} + Z_2 \bar{\psi}(i\partial\!\!\!/ - m_B)\psi$$

$$- Z_1 eQ \bar{\psi} A\!\!\!/ \psi \,, \text{ where}$$

$$m_B \equiv m - \left(1 - \frac{1}{Z_2}\right)\delta m$$

— We can make it look canonical by rescaling fields and parameters:

A^μ_B ("new") $\equiv \sqrt{Z_3}\, A^\mu$ (classical);

$\psi_B \equiv \sqrt{Z_2}\,\psi$; $e_B = e/\sqrt{Z_3}$ & $Q_B = Q \dfrac{Z_1}{Z_2}$

(For > 1 fermions, put "i" subscript on $Z_{1,2}$ – but not on Z_3 – and on Q)

$$\Rightarrow \mathcal{L}_B = -\frac{1}{4} F_{B\,\mu\nu}F^{B\,\mu\nu} + \bar{\psi}_B(i\partial\!\!\!/ - m_B)\psi_B$$

$$+ e_B Q_B \bar{\psi}_B A\!\!\!/_B \psi_B$$

(again, in general, "i" label on ψ_B, Q_B & m_B, but not on $e_B = e/\sqrt{Z_3}$)

... looks like classical, but "B" subscript everywhere: $e \to e_B$, $m \to m_B$, $Q \to Q_B$ (parameters); $\psi \to \psi_B$, $A_\mu \to A_{\mu B}$

— So, we see that observed/measured ("output") values remain eQ, m : rescaling doesn't change that) $\neq$ bare parameters ($e_B Q_B$, m_B) that we "put in" in $\mathcal{L}_B$ (i.e, $\mathcal{L}$ classical and $\mathcal{L}_{CT}$: again both are terms in "$\mathcal{L}$", that too same form, so combination taken as input) $\Rightarrow$ there is renormalization/rescaling of parameters (eQ, m vs. $e_B Q_B$, m_B)

... that too by "divergent" factor: e_B, m_B are divergent, since

e, m are of course finite/observed...

... what happened to Q ?!

Enter WT identity $\Rightarrow Z_1 = Z_2$ so that

$Q_B = Q\left(Z_1/Z_2\right) = Q$, i.e., charge is

not renormalized!

— More carefully, ratio of charges of 2 fermions is more physical, so let's consider ratio of charge of (say) electron to quark: naively, ratio of bare charges $\neq$ ratio observed, since after all $(Z_1)_e \neq (Z_1)_q$ even with only QED interaction ($Z_1 \propto Q^2$)

[Even for electron vs. muon, Z_1 due to Higgs/Yukawa coupling in loop is different...]

— Similarly, $(Z_2)_e \neq (Z_2)_q$ ...

... remarkably such that (Z_1/Z_2) same for electron, muon and quark (and equal to 1) : $(Z_1)_i = (Z_2)_i$, even if $(Z_{1,2})_i \neq (Z_{1,2})_j$ ... $\Rightarrow$

$$\frac{(Q_B)_i}{(Q_B)_j} = \frac{Q_i}{Q_j}$$ (again, this includes divergences and finite parts of $Z_{1,2}$)

— Above is another physical consequence of WT identity (earlier one was photon remaining massless even at loop-level): in fact, ratio of charges being unchanged is not "obvious" from simply gauge invariance (cf. photon being massless): indeed, Q's can be different for 2 fermions at tree-level (no principle setting value of charge, cf. photon massless already at tree-level "due to" gauge invariance) ... so, why should ratio of charges remain same at loops?! But it does so non trivially due to detailed application of WT identity

[As an aside, $(Q_B)_i/(Q_B)_j$ is not really observed, so is there a prediction here?! Maybe, e.g., if $U(1)$ gauge group (abelian) is part of a bigger/simple unified gauge group (as we will see in GUT part of course), then Q_B's are quantized so that $(Q_B)_q/(Q_B)_e$ is predicted to be $2/3$ or $1/3$ (up or down quark)]

- Compare above non-renormalization of charge to mass being renormalized : $m_B = m - \left(1 - \frac{1}{Z_2}\right)\delta m$ such that $(m_B)_e/(m_B)_q \neq \underbrace{m_e/m_q}_{\text{observed}}$

So, $\boxed{\text{summary/counting}}$ of parameters:

4 coefficients in CT's (3 amplitudes

are divergent: Σ, Γ_μ & $\Pi_{\mu\nu}$, but Σ has 2 independent divergences, i.e., with γ_μ & $\mathbb{1}$ structure):

Z_1 (for Γ_μ); Z_2 for Σ (γ_μ part); δm for $\Sigma(\mathbb{1})$ & Z_3 (for $\Pi_{\mu\nu}$) ...

... but only 2 combinations show up in renormalization (Z_1 drops out due to WT identity):

$$e_B = e/\sqrt{Z_3}\,;\; Q_B = Q\,;\; m_B = m - \left(1 - \frac{1}{Z_2}\right)\delta m$$

$\Rightarrow$ divergences "absorbed" into 2 parameters (e & m)

- Above is BPH renormalization (CL sec. 2.2)

- Instead, we can do "conventional" renormalization (CL sec. 2.1 or chapter 7 of PS), where we basically "start" with above $\mathcal{L}_B$ (no need to "add" CT: after all, CT's same form as $\mathcal{L}$ classical in earlier approach), but parameters in $\mathcal{L}_B$: e_B & m_B ($\Rightarrow$ tree-level amplitudes) are divergent ?! No worries, since divergences in loops will cancel against tree so that observed values are finite; i.e., schematically,

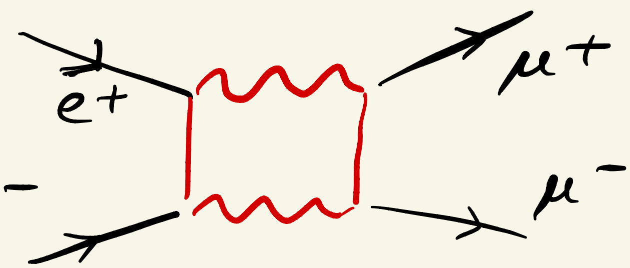
observed $\sim$ bare (e_B, m_B) (divergent)
+ loop (also divergent)
$\sim$ finite

— above "cancellation" of divergences (in 2 or 3-point functions, with $D \geqslant 0$) in conventional scheme works similarly to what we showed explicitly in BPH (in fact, just re-writing $\mathcal{L}_B$ as $\mathcal{L}_{classical} + \mathcal{L}_{CT}$

- How about other amplitudes ($D < 0$), where prediction was claimed?
- Let's do it using $\mathcal{L}_B$, e.g.

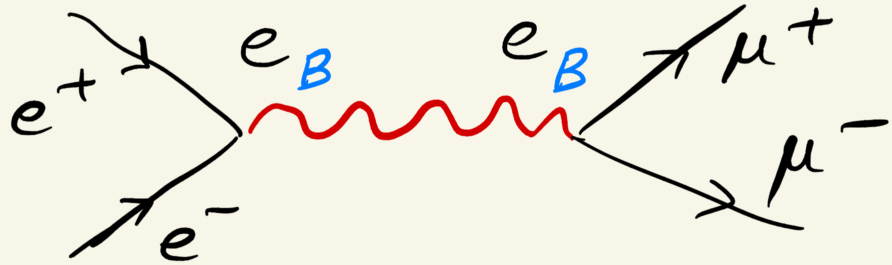
$e^+ e^- \longrightarrow \mu^+ \mu^-$ (we'll do it somewhat explicitly at 1-loop, but can be generalized to higher loops / other processes)

- Big picture first: naively (i.e., before thinking of last step of renormalization), loop contribution seems finite ($D < 0$);



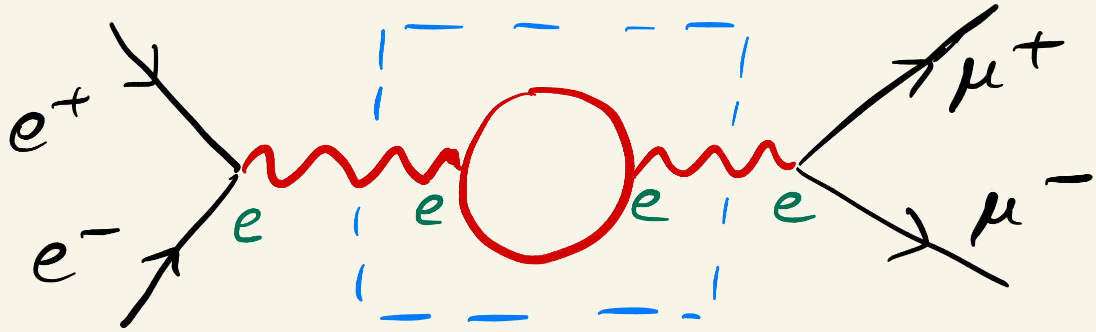
tree of course is (using $\mathcal{L}$ classical)...

... but now, i.e., using $\mathcal{L}_B$, tree-level amplitude (involving e_B) is divergent ?!



Who cancels that ?! "Must be" loop, but not above (finite)...

... instead sub-graph divergence (mentioned earlier), i.e., in $O(e^2)$ part of diagram vs. full diagram being $O(e^4)$ in amplitude



Again, even if $D < 0$ for this amplitude (earlier diagram is finite), above diagram is divergent (that's why D is superficial / naive)

— Back then, we "hand-waved" that

sub-graph divergence already encountered at earlier stage, i.e., $\Pi_{\mu\nu}$ (2-point function) tamed then ... so, no worries here (2 → 2 scattering) ...

... now, i.e., using $\mathcal{L}_B$, we see that this sub-graph (loop) divergence actually cancels e_B, so plays crucial role ...

— Next, show this explicitly, even if expected / sounds reasonable ...

... also in this process, we will identify a remnant (large) effect, i.e., running of effective coupling constant