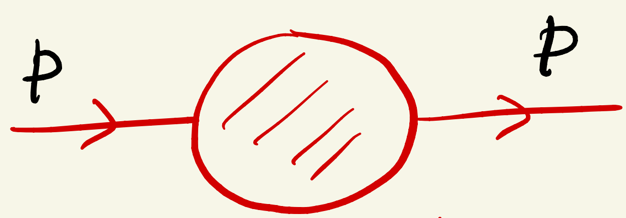
Apply $\boxed{D} = 4 - E_b - \frac{3}{2} E_f - \underbrace{\sum v_i \delta_i}_{0}$

to $\boxed{\text{QED}}$

due to $\delta_1 = [e] = 0$

- First, E_f = even by charge conservation
- So $E_f \neq 0$ starts with $E_f = 2$

(1). $\boxed{E_f = 2\ ;\ E_b = 0}$ (self-energy of electron, denoted by $\boxed{\Sigma}$: matrix in Dirac space, but no Lorentz index)



$\boxed{D = 1}$ (linear divergence)

- "matches" 1-loop estimate :

$\sim \int d^4k \, \frac{1}{k^2} \frac{1}{k} \sim \Lambda_{UV}^1$

...but valid at all loops (D independent of v_i - given $\delta_i = 0$, thus number of loops)

- However, $D = 1$ is superficial/naive
- Actual degree of divergence is

lowered by 1 to logarithmic...

... explain more later, but due to "symmetry" requiring (potentially) divergent part of $\Sigma \propto$ external momentum, p

– Crucial point: D counts powers of loop momenta "dropping" external momenta in propagators (denominator of loop integral) and in derivative coupling (numerator part)

$\Rightarrow$ if we "need" (often due to "symmetry") amplitude $\propto$ external momentum, then must get it from derivative at vertex:

$$\partial_\mu \longrightarrow \underbrace{k_\mu \text{ loop}}_{\text{for } D}, \ \boxed{p_\mu} \text{ (pick here)}$$

$\boxed{\text{or}}$ expanding propagator:

$$\frac{1}{(p-k)^2} \sim \underbrace{\frac{1}{k^2}}_{\text{for } D} + \underbrace{\frac{p}{k^3}}_{\text{"have to" pick here}} \quad (k \gg p)$$

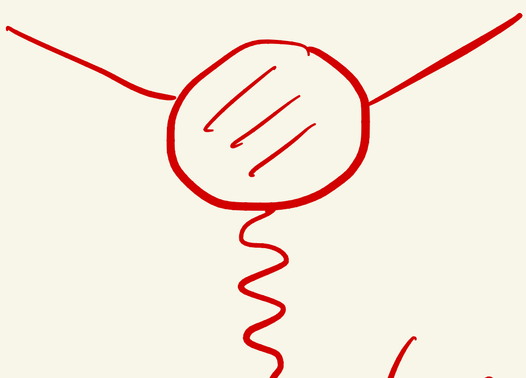
loop

external

$\Rightarrow$ more power(s) of loop momentum in propagator/denominator ... or less from derivative (vertex)/numerator

$\Rightarrow$ actual D lowered [will see explicity later, how exactly naive divergence (D) vanishes]

(2) $E_f = 2$, $E_b = 1$: vertex correction, $\boxed{\Gamma_\mu^{(\text{loop})}}$ (matrix in Dirac space)



$D = 0$ (logarithmic divergence) at all loops (matches 1 loop estimate)

- actual $D = 0$: no symmetry to reduce it (check explicitly at 1-loop later)

...larger E_b (>1) for $E_f = 2$ or $E_b = 0$, but $E_f = 4, 6 \ldots$ gives $D < 0$ (finite)

... so, left with $\boxed{E_f = 0,\ E_b \neq 0}$ to hunt for $D \geq 0$ (divergences)

- Now, A_μ is odd under charge-conjugation (C): Phys 624(?) or PS Eq. 3.149: $\bar{\Psi}\gamma_\mu\psi$ is C-odd, so is A_μ such that coupling is even

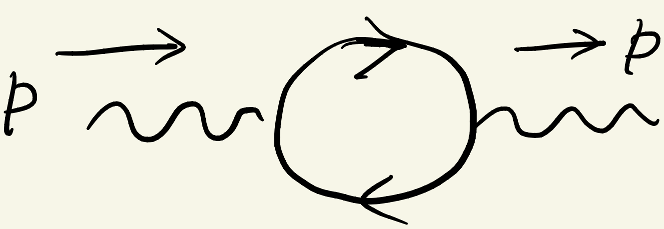
- So, E_b must be even

(3). $\boxed{E_b = 2}$ (again, $E_f = 0$):

vacuum polarization (see later why this name), denoted by $\boxed{\Pi_{\mu\nu}}$

$D = +2$ (quadratic divergence)

- matches 1-loop estimate:



$\sim \int d^4k \left(\frac{1}{k}\right)^2 \sim \Lambda_{UV}^2$

- However actual D lowered by 2

to logarithmic

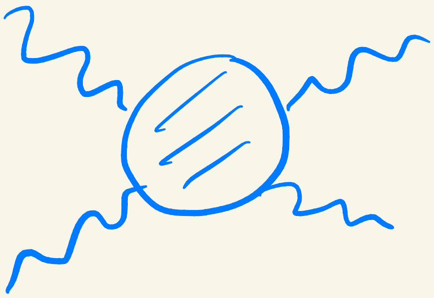
- General argument: gauge invariance
$\Rightarrow$ ($\mathcal{M}$ involving external photons) "×" momentum of external photon $= 0$ (see PS sec. 7.4)

$\Rightarrow$ $\mathcal{M}$ for n external photons only $\propto$ n powers of external momenta

[roughly, with no fermions/only photons, term "corresponding to" loop amplitude in Lagrangian $\propto$ $F_{\mu\nu}$ (only gauge-invariant form) for each photon $\Rightarrow$ amplitude $\propto$ each photon momentum]

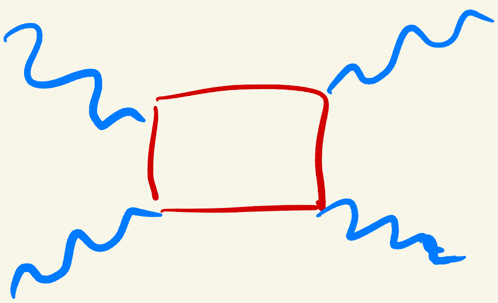
$\Rightarrow$ $\Pi_{\mu\nu} \propto [p\text{ (external)}]^2$: D lowered by 2 (will show explicitly later)

(4). $E_b = 4$ $(E_f = 0)$: 4-photon vertex (light-by-light scattering)



$D = 0$ (logarithmic divergence)

— matches 1-loop estimate:



$\sim \int d^4k \left(\frac{1}{k}\right)^4 \sim \log \Lambda_{UV}$

... but (based on above general claim) actual D lowered by 4 $\Rightarrow$ $\boxed{\text{finite}}$

— $E_b = 6, 8 \ldots$ gives $D < 0$ (finite)...

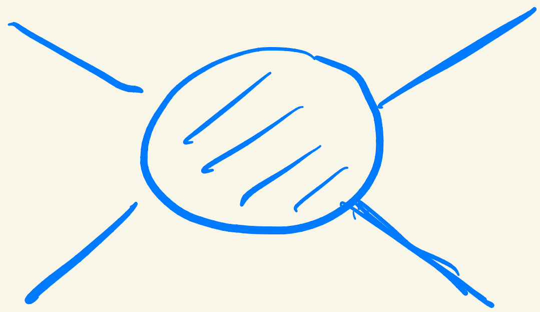
... done!

— So, 3 divergent amplitudes, but more symmetry (Ward-Takahashi identity) relates divergences in $\boxed{\Gamma_\mu}$ and $\boxed{\Sigma}$ $\Rightarrow$ in the end, can absorb ∞ into 2 free parameters

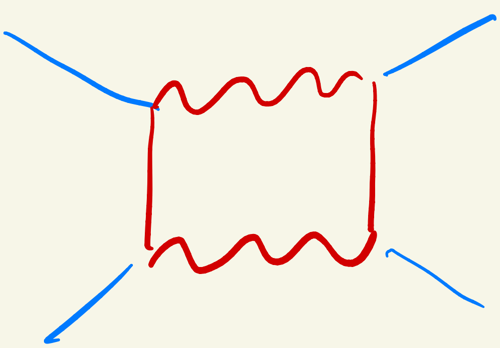
e and m, which need to be fitted to data

... then (all) other amplitudes can be predicted, e.g., $2 \to 2$ scattering of electrons / positrons has

$D = -2$ $\left(E_f = 4, E_b = 0 \right)$:



... matches 1-loop estimate:



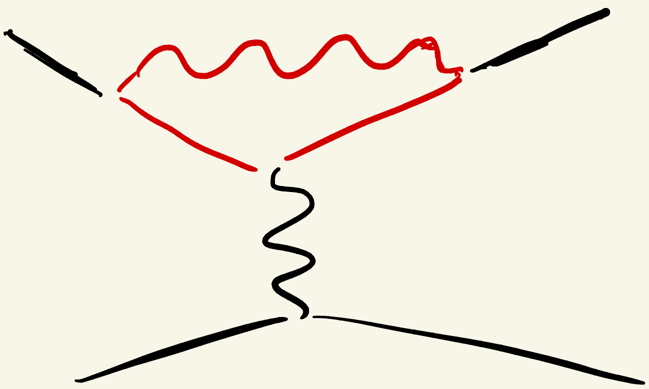
$$\sim \int d^4k \left(\frac{1}{k^2}\right)^2 \left(\frac{1}{k}\right)^2$$

$$\sim \int d^4k / k^6 \quad \text{(finite)}$$

... but ("flip" side of D being superficial / naive), degree of divergence can be "worse" than D:

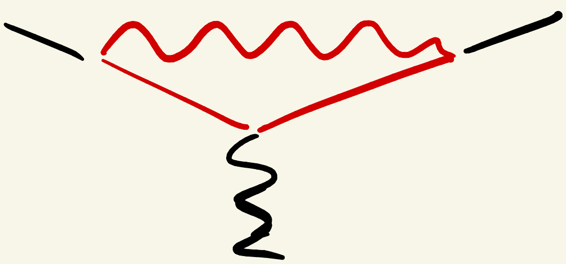
sub-diagram can be divergent,

even if full diagram is finite ($D<0$), e.g., another loop contribution to 4-fermion amplitude:



: $D=-2$ (as above)

... but



has $D=0$

However, sub-diagram divergence is "taken care of" at earlier stage of program, so same "cure" works now...

$\boxed{\text{Next}}$ topic : Ward-Takahashi (WT) identity

Motivation : aid calculation of $\Pi_{\mu\nu}$ (showing photon remains massless even at loop-level) ; by relating divergences in $\Gamma_\mu^{(loop)}$ & Σ, it will show that ratio of charges of electron & proton remains (-)1 even at loop-level

(even though proton has <u>strong</u> interactions, while electron does not)