Nambu-Goldstone boson (NGB)

from SSB of continuous (easier using radial representation)

- start with same $\mathcal{L}$, Φ:

$$\mathcal{L} = (\partial^\mu \Phi^+)(\partial_\mu \Phi) - \underbrace{\left[\mu^2 \Phi^+ \Phi + \lambda (\Phi^+ \Phi)^2\right]}_{V \,:\, \text{function of } \Phi^+ \Phi}$$

but use radial representation for Φ with VEV ($\mu^2 < 0$):

$$\Phi(x) = \frac{\left[v + \eta_r(x)\right]}{\sqrt{2}} \exp\left[i\,\theta + \zeta_r(x)/v\right]$$

radial representation

- Clearly, $\eta_r(x)$ corresponds to VEV fluctuations in "radial direction" (modulus of Φ), while $\zeta_r(x)$ is fluctuations of phase of VEV (θ) or angular direction (along trough)

– "Match" to linear represention of before (dropping quadratic & higher order in fields):

$$\Phi(x) \simeq \frac{[v + \eta_r(x)]}{\sqrt{2}} \left\{ 1 + i \left[\theta + \frac{\zeta_r(x)}{v} \right] \right\}$$

$$\simeq \frac{1}{\sqrt{2}} \left[v + \eta_r(x) + i\, \zeta_r(x) \right]$$

$\Rightarrow$ $\eta_l \simeq \eta_r$ and $\zeta_l \simeq \zeta_r$ (again, upto quadratic terms in fields)

(so, as promised, η_l & ζ_l are also – approximately – fluctuations in radial, angular directions...)

$\Rightarrow$ expect η_r to be massive, ζ_r massless: in fact, easier to see directly from radial representation by plugging above Φ form into $\mathcal{L}$: θ (constant part of phase of Φ) disappears as usual, since $\Phi \to e^{i\alpha}\Phi$ is (global) symmetry, but what about $\zeta_r(x)$: space-time dependent phase?!

- Indeed, $\zeta_r(x)$ disappears from V, which is a function of $\Phi^\dagger \Phi$, i.e., independent of phase of Φ (whether θ or ζ_r): V is invariant under local (space-time dependent) phase rotation ...

... but kinetic term: $(\partial^\mu \Phi^\dagger)(\partial_\mu \Phi)$ is only globally invariant (will become locally invariant when we couple Φ to gauge field, so keep this in mind for Higgs mechanism), again due to derivatives acting on phase, i.e., $\partial_\mu \zeta_r(x)$

$\Rightarrow$ from kinetic term for Φ, we will get terms involving $\partial_\mu \zeta_r(x)$, including kinetic term for ζ_r and $\boxed{\text{derivative}}$ interactions ...

... again no non-derivative terms for ζ_r, since it disappeared from V

$\Rightarrow \zeta_r$ is massless and (only) derivatively-coupled

– Explicitly, we get

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial^\mu \eta_r)(\partial_\mu \eta_r) + \frac{1}{2}\left(1+\frac{\eta_r}{v}\right)^2 (\partial^\mu \zeta_r)(\partial_\mu \zeta_r)}_{\text{from } (\partial^\mu \Phi^+)(\partial_\mu \Phi)}$$

$$- (\text{from } V) \underbrace{\left[+\frac{1}{2}(v+\eta)^2 \mu^2 + \frac{1}{4}\lambda (v+\eta)^4\right]}$$

$$= \eta^2\Big(+\frac{1}{2}\underbrace{\mu^2}_{<0\,!} + \frac{1}{4}\lambda v^2\, 6\Big) +$$

$$\eta^1\left(+\frac{1}{2} v \cdot 2\, \mu^2 + \frac{1}{4}\lambda v^3 4\right) \;(!)$$

$$+ \;\eta^3, \eta^4 \text{ terms}$$

$$= \boxed{+\frac{1}{2} m_\eta^2\, \eta^2} + \text{no } \eta^1 + \eta^3, \eta^4$$

where $m_\eta^2 = 2\lambda v^2$

... so, explicitly "matches" linear representation manifestly at quadratic order in fields (η_r massive, ζ_r massless; no η_r^1 ...)

- Also, derivative interactions only for ζ_r (as expected)
- However, ζ_r interactions look non-renormalizable ([couplings] < 0)...

...but we know from linear representation that theory is renormalizable!

- Resolution: there are two ζ_r couplings, with coefficients being related (again, remnant of symmetry):

$$\eta_r^1 (\partial^\mu \zeta_r)(\partial_\mu \zeta_r) \text{ and } \eta_r^2 (\partial^\mu \zeta_r)(\partial_\mu \zeta_r)$$

...so "must be" that non-renormalizable effects "cancel" between these 2: looks like a "conspiracy", but is guaranteed by residual symmetry (cf. for a general scalar field with such derivative interactions, i.e., with no relation between couplings, we will not get this "miracle"!)

Summary of tale of 2 representations (linear & radial here... but will be a recurring theme)

— Physical observable, e.g., scattering amplitudes, independent of choice of representation used to compute

— And, certain results easier to see in one representation vs. another ...

... so "exploit" this feature!

— In above example, we find

(1). Amplitudes involving NGB (massless scalar from continuous SSB) $\propto$ NGB momenta

$\Rightarrow$ suppressed at low energies

— obvious in radial representation, since $\mathcal{L}_r$ terms in Lagrangian involve derivatives in position-space (momenta upon Fourier transforming)

— "obscure" in linear representation: $\mathcal{L}_\ell$ does have non-derivative couplings ... but we know amplitudes $\propto$ NGB momenta from other viewpoint and physics is representation-independent $\Rightarrow$ in linear representation, it "must be" that there are multiple contributions (or diagrams) for an amplitude, each of which is not manifestly $\propto$ NGB momentum (since from non-derivative couplings)... but combine in such a way that net result is $\propto$ NGB momenta (and exactly same as radial representation)...

...note that relations between various non-derivative couplings (again, residual of symmetry, no tuning) crucial for this "miracle" to happen...

See HW 3.3 & 3.4 for examples of representation independence & NGB scattering amplitude $\propto$ momenta

$\Rightarrow$ for explicit symmetry breaking, where couplings are not related, we will not get such a result (even if we choose scalar to be massless)

(2). Renormalizability: obvious in linear representation, but "hidden" in radial representation: subtle cancellations between "individually" non-renormalizable effects, again a result of surviving symmetry relating them...

$\Rightarrow$ general lesson: in representation where certain result is not manifest, its crucial to use relations among different effects/couplings which are enforced by (remnant) symmetry

[No mass for NGB: obvious from radial representation, but not so difficult to find in linear representation also, cf. "effectively" derivative coupling in linear representation or renormalizability in radial representation, as discussed above ...]

— Onto $\boxed{\text{Goldstone's theorem}}$: for every spontaneously broken (global) continuous symmetry, there exists massless scalar particle (NGB)

— Here, general proof (vs. specific simple case studied above) for scalar field theories (i.e., SSB from scalar VEV's) at classical-level (NGB cannot acquire mass from quantum corrections, since those respect same symmetries)

— More general / different proof (i.e., not assuming from scalar VEV) in LP 13.5.1 or CL sec. 5.3 (p. 144)

SSB by $\boxed{\text{multiple}}$ scalar fields

- Goals: (i). show massless scalar for every SSB (global) and (ii). identify combination of scalar fields which is NGB (based on PS sec 11.1)

- Start with
$\mathcal{L}$ = derivative (kinetic) terms $- V(\phi^a)$,

non-derivative potential

where $a = 1, 2 \ldots$ are real scalar fields (e.g., re-write complex scalar fields in terms of real & imaginary components)

- As usual, minimize V by x_μ-independent (constant) ϕ_0^a, where "0" subscript denotes VEV (just to avoid clutter with using $\langle \phi \rangle$)

- Expand V about minimum:

$$V(\phi) \simeq V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^a (\phi - \phi_0)^b m^2_{ab},$$

where 1^{st} term, $V(\phi_0)$ is field-independent

and $m^2_{ab} = \left.\dfrac{\partial^2 V}{\partial\phi^a \partial\phi^b}\right|_{\phi_0}$ ← VEV is a (symmetric

in field space) matrix whose eigenvalues give masses; other terms in $V(\phi)$, i.e., cubic and higher powers in $(\phi - \phi_0)$ are interactions for fluctuations around minimum

[no term linear in $(\phi - \phi_0)$, since ϕ_0 is minimum, i.e., $\partial V / \partial\phi^a \big|_{\phi_0} = 0$]

- Since ϕ_0 is minimum, eignenvalues of $m^2_{ab} \geqslant 0$

- $\boxed{\text{Goal}}$: for every symmetry which is spontaneously broken, there is a zero eigenvalue of m^2_{ab}; corresponding eigenvector in field space is NGB

- Now, a continuous transformation (not yet a symmetry) has infinitesimal version:

shift in $\phi^a \equiv \alpha$ (small) $\times$ function of all fields

$= \alpha \times \Delta^a(\phi)$ (in general, depends on a)

e.g., U(1)/abelian symmetry for complex $\Phi \equiv \phi^{a=1} + i\phi^{a=2}$, we have shift in $\Phi = i\alpha\Phi$

$$= \alpha\left(i\phi^1 - \phi^2\right)$$

$$\Rightarrow \Delta^1 = -\phi^2 \, ; \; \Delta^2 = \phi_1$$

- Such a transformation is symmetry if $\mathcal{L}$ is unchanged (or shifts by total derivative)
- choose fields to be x_μ-independent (constant values) - but not yet VEV (ϕ_0), so derivative terms in $\mathcal{L}$ vanish

$\Rightarrow V(\phi\,[\equiv \text{constant}])$ by itself is invariant under symmetry transformation:

$$V(\phi^a) = V\left[\phi^a + \alpha\, \Delta^a(\phi)\right]$$

$$\simeq V(\phi^a) + \alpha \sum_a \Delta^a(\phi)\, \partial V / \partial \phi^a$$

$$\Rightarrow \sum_a \Delta^a(\phi)\, \partial V / \partial \phi^a = 0 \quad (\text{use } \alpha \text{ small})$$

(again, ϕ^a = constant, but need not be $\phi^a_0 \leftarrow$ VEV)

Taking another derivative with respect to ϕ^b ... then set ϕ^a to VEV:

$$0 = \sum_a \left(\frac{\partial \Delta^a}{\partial \phi^b}\right)\Bigg|_{\phi_0} \underbrace{\frac{\partial V}{\partial \phi^a}\Bigg|_{\phi_0}}_{\substack{=0 \text{ since} \\ \phi_0 \text{ is minimum}}} + \sum_a \Delta^a(\phi_0)\, \underbrace{m^2_{ab}}_{\frac{\partial^2 V}{\partial \phi^a \partial \phi_b}\big|_{\phi_0}}$$

i.e., $\sum_a m^2_{ba}\, \Delta^a(\phi_0) = 0$

or $(m^2\text{- matrix}) \cdot [\Delta(\phi_0) \text{ column-vector}] = 0$

Two cases

(1). Symmetry unbroken in ground state/vacuum:

ϕ_0 is invariant under transformation $\Rightarrow$

$\Delta^a(\phi_0) = 0$ for all a, so above matrix equation trivially satisfield

(2). Spontaneously broken symmetry: ϕ_0 is not invariant under transformation

$\Rightarrow \Delta^a(\phi_0) \neq 0$ for at least some a

$\Rightarrow \Delta^a(\phi_0)$ is (non-trivial) eigenvector in field space with zero eigenvalue of m^2_{ba}: again

$(m^2\text{-matrix}) \cdot [\Delta(\phi_0) \text{ column-vector}] = 0$

$\neq$ null vector

– To be explicit, component of eigenvector along $(\phi^a - \phi^a_0)$ is $\Delta^a(\phi_0)$, i.e.,

$$NGB = \sum_a \underbrace{\Delta^a(\phi_0)}_{\text{constant}} \underbrace{(\phi^a - \phi^a_0)}_{\text{fluctuation around VEV}}$$

- So, NGB has "overlap" with only those fields (fluctuations around VEV) which transform inhomogeneously in ground state, i.e., $\Delta^a(\phi_0) \neq 0$

- Note that transformed (constant) field configuration: $\phi_0^a + \underbrace{\Delta^a(\phi_0)}_{\neq 0 \text{ for some } a\text{'s}} \alpha$

also minimizes energy (like ϕ_0^a itself) due to symmetry, i.e.,

$$V(\phi_0^a) = V\left[\phi_0^a + \alpha\, \Delta^a(\phi_0)\right]$$ (particular case of general above)

$\Rightarrow$ vacuum not unique

e.g., U(1)/abelian symmetry: setting phase of VEV $= 0$, we have in earlier notation (linear representation)

$$\Phi = \frac{1}{\sqrt{2}}\Big(\underbrace{v + \eta}_{\phi^1} + i\underbrace{\xi}_{\phi^2}\Big) \quad \begin{pmatrix} 1 = \eta \\ 2 = \xi \end{pmatrix}$$

(η, ζ are fluctuations around VEU, i.e., with 0 VEU themselves)

- Now under $\Phi \to e^{i\alpha}\Phi$ with α small, we have

$$\eta \to \eta - \zeta \quad \text{and} \quad \zeta \to \zeta + (v + \eta)$$

i.e., $\overset{\leftarrow \eta}{\Delta^1}(\phi_0) = 0$, while $\overset{\leftarrow \zeta}{\Delta^2}(\phi_0) = v \; (\neq 0)$

$\Rightarrow$ NGB vector in $\phi^{1,2}$ (or η, ζ) space $\propto \begin{pmatrix} 0 = \Delta^1(\phi_0) \\ 1 = \Delta^2(\phi_0) \end{pmatrix}$, i.e.,

ζ (ϕ^2) is massless (in agreement with earlier calculation): again, ζ transforms inhomogeneously ("into" v) in ground state