Higgs mechanism: general / R_ξ-gauges (renormalizable)

- Go back to linear representation for Φ:

$$\Phi(x) = \left[v + \eta_\ell(x) + i\,\zeta_\ell(x) \right] / \sqrt{2}$$ (setting phase of VEV, $\theta = 0$)

[again, "keep" $\zeta_\ell(x)$, even though we know from radial representation/unitarity gauge that it is unphysical.]

- Plug above Φ in $\mathcal{L}$ (drop ψ):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu \eta_\ell - e Q_\phi A_\mu \zeta_\ell\right)^2$$
$$+ \frac{1}{2}\left[\partial_\mu \zeta_\ell + e Q_\phi A_\mu (v + \eta_\ell)\right]^2 - V(\Phi)$$

- As for global symmetry case, $V(\Phi)$ will give mass term for η_ℓ, but not for ζ_ℓ
- Also, 1st term in 2nd line will give mass term for A_μ (like with radial representation discussed earlier)
- So, focus on "new" terms:

(1). a "mixed" gauge coupling:

$(-\partial_\mu \eta_\ell A_\mu \zeta_\ell + \partial_\mu \zeta_\ell A_\mu \eta) e Q_\phi$ from 2nd term in 1st line and 1st term in 2nd line (relevant for calculating amplitudes in R_ξ-gauge: see HW 4.2.2)

(2). "Mixed" quadratic in fields term from 1st term in 2nd line:

$\partial_\mu \zeta_\ell A^\mu \underbrace{e Q_\phi v}_{M_A}$, ... $\Rightarrow$ can we get well-defined A_μ, ζ_ℓ propagators?!

... No, i.e., need to "diagonalize"...

Relatedly/more formally, even though gauge boson has mass term, $\mathcal{L}$ still has a gauge symmetry [due to "presence" of $\zeta_\ell(x)$, cf. unitarity gauge, where gauge invariance was lost]:

$\eta_\ell \to \eta_\ell - e Q_\phi \alpha(x) \zeta_\ell$; $\zeta_\ell \to \zeta_\ell + e Q_\phi \alpha(v+\eta)$

and $A_\mu \to A_\mu - \partial_\mu \alpha$

- Explicitly, gauge boson mass term – by itself – is (obviously) not invariant:

$$\underbrace{M_A^2}_{e^2 v^2 Q_\phi^2} A_\mu^2 \longrightarrow M_A^2 \left[A_\mu^2 + \boxed{A_\mu \partial^\mu \alpha} + \mathcal{O}(\alpha^2) \right]$$

cancel

... who cancels it ?!

... shift of mixed term above:

$$\underbrace{M_A}_{e v Q_\phi} \partial_\mu \zeta A^\mu \longrightarrow M_A \left[\partial_\mu \zeta A^\mu + \boxed{\overbrace{e Q_\phi v}^{M_A} \partial_\mu \alpha A^\mu} \right]$$

from ∂_μ(shift in ζ): $\partial_\mu \alpha \; e v Q_\phi$

- Bottomline: we need to add gauge-fixing term (like for massless photon, where also propagator ill-defined "due to" gauge invariance): choose

$$\underbrace{\mathcal{L}_{GF}}_{\text{gauge-fixing}} = -\frac{1}{2\xi} \left(\partial_\mu A^\mu - \xi M_A \zeta_\ell \right)^2$$

so that mixed $(A_\mu - \zeta_\ell)$ terms from $\mathcal{L}_{GF}$ and earlier $\mathcal{L}$ combine to form total derivative, thus drop out at level of action, i.e., reduce to "surface integral" $\to 0$, assuming fields die-off fast

$$\Rightarrow \mathcal{L}^{(\zeta_\ell)}_{free} = \frac{1}{2}\left[(\partial^\mu \zeta_\ell)(\partial_\mu \zeta_\ell) - \xi^2 M_A^2 \zeta_\ell^2\right]$$

from $\boxed{\mathcal{L}_{GF}}$

$$\mathcal{L}^{(A_\mu)}_{free} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} M_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$\Rightarrow$ we get propagators (HW 4):

(gauge boson): $$i D_{\mu\nu}(k) = \frac{-\left[g_{\mu\nu} - \frac{(1-\xi)\, k_\mu k_\nu}{k^2 - \xi M_A^2}\right]}{k^2 - M_A^2 + i\varepsilon}$$

("would-be" NGB): $$i\Delta(k) = \frac{i}{k^2 - \xi M_A^2}$$

(of course, η_ℓ unaffected by $\mathcal{L}_{GF}$)

Let's interpret/digest above result

— For fixed/given ξ, we see that gauge boson propagator in UV limit, i.e., $k \gg M_A$ is "well-behaved":

$$D_{\mu\nu}(k) \longrightarrow -\left[g_{\mu\nu} - (1-\xi)k_\mu k_\nu / k^2\right] / k^2,$$

indeed same as for massless case (including $k_\mu k_\nu$ piece) [basically $O(1/k^2)$]!

$\Rightarrow$ $\boxed{\text{Higgs model is manifestly renormalizable in } R_\xi\text{-gauge}}$

(cf. unitarity gauge, where it instead scaled as $\sim 1/M_A^2$ in UV-limit, due to $k_\mu k_\nu$ piece..., raising spectre of non-renormalizability.
more on "connection" of R_ξ to unitarity gauge is discussed below)

— Also, R_ξ-gauge provides a consistent quantum realization (no worry of quantum corrections "taking us out of" unitarity gauge)

... but not so fast (this might be too good to be true: well it is!)...

Analysing R_ξ-gauge

(1). Propagators of A_μ & φ_ℓ depend on ξ, thus so might amplitudes...

... but we "know" that ξ is an unphysical parameter, so in the end, amplitudes must be independent of ξ (just like for case of massless photon, where propagator is ξ-dependent), i.e., different ξ describe same physics (we will illustrate this via example later)...

(2). Related to above, R_ξ-gauge comes with "price" of including actually unphysical d.o.f. (2 of them):

(a). "would-be" (since we know from unitarity gauge that it is unphysical, i.e., can be removed) NGB, φ_ℓ:

even without "foresight" from unitary gauge, i.e., starting in R_ξ-gauge itself, we see that "mass" of ξ_ℓ (pole of that propagator) is dependent on ξ (again, an unphysical parameter)

(b). similarly, gauge boson propagator ($k_\mu k_\nu$ part) being ξ-dependent is "symptomatic" of it including unphysical polarizations (again, as for massless photon propagator: ξ-dependence in that case is "due to" presence of longitudinal & time-like or scalar polarizations)...

Equivalently, R_ξ-gauge propagator for gauge boson $\neq$ that in unitarity gauge (latter incorporates all / only physical polarizations: 2 transverse + 1 longitudinal for massive case):

in fact, R_ξ-gauge propagator has additional (fictitious) "pole" at ξM_A^2!

$\Rightarrow$ R_ξ-gauge contains also time-like or scalar polarization of gauge boson

— Needless to say, above 2 unphysical d.o.f. (again, would-be NGB & scalar or time-like polarization of gauge boson) cannot appear as external lines in Feynman diagrams, but are needed as internal lines (part of propagators)

— Also, we expect ξ-dependence dropping out of net amplitude for a physical process as a result of cancellation between these 2 unphysical effects

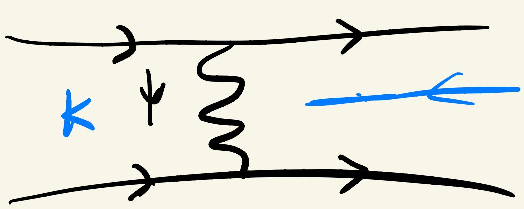
(3). What about unitary gauge: is it renormalizable ?!

(a). Indeed, $\xi \to \infty$ limit of R_ξ seems to reproduce unitary gauge, since would-be NGB becomes infinitely heavy (propagator $\to 0$), while gauge boson propagator as in unitarity gauge (b). Combining above limit with R_ξ-gauge being clearly renormalizable and physical amplitudes being ξ-independent, we see that unitarity gauge is also renormalizable, albeit "secretly/subtly" so, i.e., order of steps is crucial here: again, first calculate loop diagrams for fixed, finite ξ, where divergences are under control (renormalizable) due to gauge boson propagator being "well-behaved"... only later take $\xi \to \infty$ to get to unitarity gauge. [Instead, if we start with unitarity gauge, i.e., first take $\xi \to \infty$, then will not "see" renormalizability, since propagator seems to "misbehave..."]

— Next, indicate how ξ-dependence cancels via example (for general discussion, see PS)

ξ - independence of fermion-fermion scattering in massive gauge boson theory (Higgs model)

Case (1): fermion couples vectorially to gauge boson; i.e., L, R chiralities have same charge (set to 1 without loss of generality) $\Rightarrow$ mass term for fermion: $m\bar{\psi}\psi = m\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right)$ is gauge invariant (net charge = 0) and for generic charge of scalar (Q_ϕ), we cannot couple ψ to ϕ (at renormalizable level) in a gauge-invariant manner $\Rightarrow$ ψ does not interact with either η_ℓ or ξ_ℓ $\Rightarrow$ fermion-fermion scattering only from gauge boson exchange:



ξ - dependence in propagator

– Now, ξ - dependent piece of gauge boson propagator $\propto k_\mu k_\nu \ldots \Rightarrow$ that part of coupling $\sim k_\mu j^\mu_V$

$\longrightarrow \partial_\mu j^\mu_V = 0$ (as argued earlier, i.e., with explicit mass term for gauge boson) $\Rightarrow$ no ξ - dependence in amplitude

More interesting/sublte is case (2) with, say, only ψ_L has charge (= 1), while ψ_R has zero charge $\Rightarrow m_\psi$ must come from ψ coupling to Φ with $Q_\phi = 1$, i.e.,

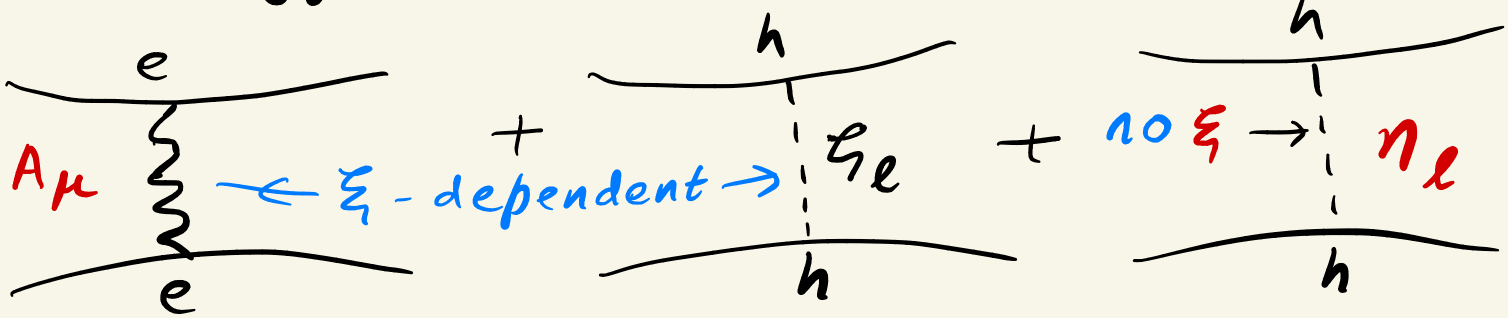
$$\mathcal{L} \ni -h\left(\bar{\psi}_L \, \Phi \, \psi_R + \bar{\psi}_R \, \Phi^* \psi_L\right)$$

charges -1 $\;+1\;$ 0
$\Rightarrow$ total $= 0$

giving (after SSB): $m_\psi = h\, v/\sqrt{2}$
(again, bare/explicit mass term for ψ is not gauge invariant ... like for A_μ)

$\Rightarrow$ $\boxed{\text{fermion couples to } \eta_\ell, \xi_\ell \text{ (from } \Phi) \sim h \sim m_\psi / v}$

So, we have 3 contributions to fermion-fermion scattering: exchanges of (massive) gauge boson (A_μ); would-be NGB (η_ℓ) & Higgs boson (η)



- we'll sketch main points here: for detailed calculation, see later note, based on PS

(1). Since we want to only keep track of ξ - dependence here (in order to see how it cancels in the end), we can "drop" η_ℓ exchange (which is manifestly ξ - independent: both coupling of $\psi - \eta_\ell$ and η_ℓ propagator)

$\Rightarrow$ ξ - dependence cancels between A_μ & ξ_ℓ exchanges

(2). Now, ξ_ℓ-ψ coupling $\sim h \sim m_\psi / v$, whereas A_μ-ψ coupling $\sim e$ ... so, how can the (ξ-dependences between) 2 cancel each other ?!

(3). Crucially, ξ-dependence in A_μ propagator is only in $k_\mu k_\nu$ piece, which does contribute in this case (cf. earlier) due to chiral coupling of ψ, i.e., $\propto \partial_\mu j^\mu_A \propto m_\psi$ ... like G_ℓ exchange !

So, at least couplings (effectively, but partly) "match" between G_ℓ exchange & "relevant" (ξ-dependent) part of A_μ exchange ...

... but what about "rest" of propagators (& "e" in coupling of A_μ exchange) ?!

(4). Getting into more details of G_ℓ and A_μ exchanges, we see that

(a) η_ℓ exchange part of amplitude (schematically, i.e., dropping factors of i, 2 etc & Dirac structure) ~

$$\underset{\substack{\uparrow \\ \text{"due to"} \\ \Phi \ni i\eta_\ell}}{(\bar{u}\gamma_5 u)^2} \underbrace{\left(\frac{m_\psi}{v}\right)^2}_{\sim h^2 \text{ (coupling)}} \underbrace{\frac{1}{k^2 - \xi M_A^2}}_{\eta_\ell \text{ propagator}}$$

(b) A_μ propagator can be split into ~ (i) $\dfrac{\left[g_{\mu\nu} - \dfrac{k_\mu k_\nu}{M_A^2}\right]}{k^2 - M_A^2}$, which is

ξ-independent: indeed it is unitarity gauge propagator $\propto$ sum over 3 physical polarizations ...and

(ii) ξ-dependent piece ~ $\left(k_\mu k_\nu / M_A^2\right) \dfrac{1}{\left(k^2 - \xi M_A^2\right)}$ } must be time-like or spatial polarization

So, A_μ-exchange part of amplitude ~

$$\underset{\substack{\uparrow \\ \text{coupling}}}{e^2} \frac{(\bar{u} k_\mu \gamma^\mu \gamma_5 u)(\bar{u} k_\nu \gamma^\nu \gamma_5 u)}{M_A^2 (k^2 - \xi M_A^2)}, \text{ with } \left(\begin{array}{l}\gamma_5 \text{ due to} \\ \text{chiral} \\ \text{coupling}\end{array}\right)$$

$\bar{u} k_\mu \gamma^\mu \gamma_5 u \sim m_\psi \bar{u} \gamma_5 u$ (again, based on $\underbrace{k_\mu j^\mu_A}_{\bar{u} \gamma^\mu \gamma_5 u} \rightarrow \partial_\mu j^\mu_A \sim m_\psi \bar{\Psi} \gamma_5 \psi \ldots$)

$\Rightarrow$ using also $e^2 v^2 \sim M_A^2$, we see that ξ-dependent A_μ exchange $\sim$

$$(m_\psi / v)^2 \frac{(\bar{u} \gamma_5 u)^2}{k^2 - \xi M_A^2},$$

i.e., (e cancels)

contribution of time-like or scalar polarization can (and indeed does) cancel G_ℓ-exchange!

$\Rightarrow$ as expected, in net amplitude, 2 unphysical effects (again, G_ℓ and scalar/time-like gauge boson polarization) cancel each other, thereby also removing ξ-dependence

$\Rightarrow$ only contribution of physical d.o.f. remains (η & 3 gauge boson polarizations)