Spontaneous breaking of $\boxed{\text{Gauge}}$ Symmetry: $\boxed{\text{Higgs}}$ mechanism:

Basic idea

Outline

— gauge earlier continuous [U(1)] global symmetry (no explicit mass term for gauge field: $\mathcal{L}$ is gauge-invariant) $\Rightarrow$

- NGB of global case "disappears"
- gauge boson massive
- other massive scalar (radial mode) of global case survives

— above features easily seen in "unitarity" gauge (where only physical d.o.f. are kept)

— However, in unitarity gauge, (massive) gauge boson propagator has same form as with explicit mass term...

... so, $\boxed{\text{naively}}$ renormalizability aspect not modified (that was entire motivation for SSB) ?!

— Not quite, since there's additional particle (Higgs boson) — "leftover" from SSB vs. explicit mass term ⇒ 2 models are not identical ... ⇒ suggests renormalizability can be different (can hope SSB is renormalizable even if explicit mass term is not)

— Renormalizability more transparent in different representation (general or R_ξ-gauge): can show Higgs mechanism / SSB is always renormalizable, vs. explicit mass term can be non-renormalizable ...

... again, even though end result — as far as gauge boson is concerned — looks same, presence of Higgs boson in SSB makes a difference

Unitarity gauge

- Gauge continuous global symmetry of earlier complex scalar model: couple Φ (with charge Q_ϕ) to gauge field A_μ; add Dirac fermion (ψ) (with charge Q_ψ set to 1, without loss of generality) also coupled vectorially to gauge field:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi)$$
$$+ \bar{\psi}(i \not{D} - m_\psi)\psi$$

bare mass allowed
cf. later with chiral coupling

where $D_\mu \Phi = (\partial_\mu + ie\, Q_\phi A_\mu)\Phi$,

$D_\mu \psi = (\partial_\mu + ie\, \underset{Q_\psi}{1} A_\mu)\psi$ and

$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2 \quad (\mu^2 < 0)$

[For generic Q_ϕ, a coupling of ψ to Φ will not be gauge-invariant, thus is forbidden.]

$\mathcal{L}$ is invariant under local phase rotations on Φ (and ψ), cf. global earlier: this gauge transformation is

$$\Phi(x) \to \exp[-ieQ_\phi \alpha(x)]\,\Phi \,;\; A_\mu \to A_\mu + \partial_\mu \alpha$$

and $\psi \to \exp[-ie\,\alpha(x)]\,\psi$

– choose radial representation for Φ:

$$\Phi = \frac{[v + \eta_r(x)]}{\sqrt{2}} \exp\left\{ i \left[\theta + \frac{\xi_r(x)}{v} \right] \right\}$$

– As before (i.e., global case), ξ_r disappears from $V(\Phi)$, but what about kinetic or derivative terms in $\mathcal{L}$?!

– Recall that in global symmetry mode, $\mathcal{L}_{\text{kinetic}}$ was not invariant under local phase rotations (even if potential was) $\Rightarrow$ $\xi_r(x)$ remained in derivate terms (kinetic & interactions)...

... but now (in gauged model), kinetic

part of $\mathcal{L}$ (i.e, entire $\mathcal{L}$) is locally $U(1)$-invariant $\Rightarrow$ space-time dependent part of phase of Φ, i.e., $\zeta_r(x)$, should be removable by a suitable gauge transformation...

— Indeed, choosing $-eQ_\phi \underbrace{\alpha(x)}_{\text{gauge transformation parameter}} = \overset{\text{constant}}{\theta} + \underbrace{\zeta_r(x)/v}_{\text{field}}$,

unitarity gauge

we see that $\zeta_r(x)$ disappears completely (again, even from derivative terms)...

... but there is no "loss" of d.o.f. since gauge boson (which was massless "before" coupling to Φ VEV) now becomes massive, thus acquiring an extra d.o.f. (polarization) as follows.

[Gauge boson mass, i.e., breaking of gauge invariance is "expected", since we cannot do a further local phase rotation while remaining in unitarity gauge, i.e., without $\zeta_r(x)$ ... so, no gauge invariance...]

— Plug above Φ [with gauge choice of no $\zeta_r(x)$] into $\mathcal{L}$:

$$\mathcal{L} \ni (D^\mu \Phi)^\dagger (D_\mu \Phi) = \frac{M_A^2}{2} A_\mu A^\mu + \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta),$$

$$+ \; e^2 Q_\phi^2 \, A^\mu A_\mu \, \eta^2 \Big\} \text{ interaction}$$

where $\boxed{M_A^2 = e^2 v^2 Q_\phi^2}$ is mass term for gauge field, of correct sign (same as in $\mathcal{L}_{\text{Proca}}$, i.e., explicit mass term): again $A_\mu A^\mu = (A^0)^2 - (A^i)^2$ so that $\mathcal{L} \ni -\frac{1}{2} M_A^2 (A^i)^2$ or $V = +\frac{M_A^2}{2}(A^i)^2$ (i.e., mass^2 in potential > 0 for "physical," space-like components of gauge field)...

... but massive gauge field has three polarizations (2 transverse + 1 longitudianal: see note below) vs. only 2 transverse for massless gauge boson (again, before "turning-on" coupling A_μ to Φ VEV)

$\Rightarrow$ "conservation" of d.o.f. : what's "lost" by scalar/spin-0 sector (i.e., $\xi_r(x)$ disappearing) is gained by gauge field/spin-1 sector (i.e., longitudinal polarization...

[Gauge field "eats" NGB to become massive, while (massive) η_r "untouched" in this process, i.e., is a physical d.o.f. in both global and gauged models: η is called the Higgs boson (more on its "role" in a bit]

– Just to emphasize this mechanism, suppose we "start" with gauged model (with physical d.o.f. being η and three A_μ polarizations), but then take the $\boxed{\text{limit } Q_\phi \to 0}$, i.e., scalar "decouples" from A_μ (which however "continues" to couple to ψ): phase rotation on Φ (but not on ψ) becomes ("goes back to") global symmetry

$\Rightarrow \zeta_r(x)$ cannot be removed, thus it "re-appears" (physically), but then A_μ "loses" longitudinal polarization, i.e., (massive) η + (2 transverse and 1 longitudinal polarizations of A_μ (massive)

$\xrightarrow{Q_\phi \to 0}$ (massive) η (no change), massless ζ + 2 transverse polarizations of A_μ (massless)

(of course, ψ is "spectator" to this process)

- In fact, we have Goldstone's equivalence theorem (based on above intuition: for "proof" see PS sec. 21.2 and examples in HW 4,10): amplitude for emission or absorption of longitudinal polarization of massive gauge boson in high-energy limit (E of gauge boson $\gg M_A$) become (approximately) equal to that of corresponding ("eaten") NGB
- Next, (elephant in the room!) is Higgs mechanism renormalizable?!

... first, look into polarization vectors ...

Massive gauge boson polarization vectors

– More physically / intuitively to begin with, start in rest-frame of gauge boson (cannot do it for massless case!), where spin (1) fixes direction, say $+z$

– if boost to go to a general frame is along / opposite to spin ($\pm z$ here), then we get helicity $= \pm 1$ (transverse polarizations: $\epsilon_\mu \perp \bar{k}$, where ϵ_μ is polarization vector and $\bar{k}$ is gauge boson momentum in general frame; note spin is along / opposite to $\bar{k}$)

– if boost is in plane $\perp$ spin, then helicity $= 0$ (longitudinal polarization: $\epsilon_\mu \parallel \bar{k}$; note spin $\perp \bar{k}$)

– More precisely / mathematically, divergence of EOM for A_μ from Proca Lagrangian (explicit gauge boson mass term) gives

$\partial_\mu A^\mu = 0$ (see Mandl, Shaw sec. 11.3 or Ryder sec. 4.5), i.e., "automatically", cf. Maxwell Lagrangian (massless gauge field) where this is "imposed" as a subsidiary (or gauge fixing) condition

$\Rightarrow \boxed{k_\mu \cdot \epsilon^\mu = 0}$ (k_μ, ϵ_μ are gauge boson momentum, polarization vector in general frame)

– To determine ϵ^μ, go to rest-frame of gauge boson (cannot for massless case):

$k_\mu^{rest} = (M_A, \bar{0}) \Rightarrow \epsilon_\mu$ must be purely spatial

$\Rightarrow$ three possible ϵ_μ's, say, $\hat{x}, \hat{y}, \hat{z}$

– Then, boost along $\hat{k}$ to go to general frame, i.e., $\boxed{k_\mu = \left(\sqrt{M_A^2 + |\bar{k}|^2},\ \bar{k}\right)}$:

two $\bar{\epsilon}$'s (again, purely spatial) of rest frame which are $\perp$ to $\hat{k}$ (boost) are unchanged (still purely spatial: transverse (T) polarizations, since

$\bar{\epsilon}_T \cdot \bar{k} = 0$, i.e., $\bar{\epsilon}_T$ in plane $\perp \bar{k}$)...

whereas 3rd $\bar{\epsilon}$ in rest-frame is along/opposite to boost, thus is transformed into (in general frame):

$$\epsilon_\mu^{L \text{(longitudinal)}} = \underbrace{1/M_A}_{\text{fixed by normalization}} \cdot \Big(\underbrace{|\bar{k}|}_{\text{fixed by } k^\mu \cdot \epsilon_\mu = 0}, \; E \hat{k} \Big)$$

boost direction

(longitudinal polarization, since spatial part $\parallel \bar{k}$ ... but there is also time-like part in general frame)

– Clearly (in general frame), we have

$$\underbrace{\epsilon_L^\mu}_{\substack{\text{spatial} \\ \text{part} \parallel \bar{k}}} \cdot \underbrace{\epsilon_\mu^T}_{\substack{\text{purely spatial,} \\ \perp \bar{k}}} = 0$$

– Note in high-energy limit ($E \gg M_A$, as relevant for equivalence theorem), we get $\epsilon_\mu^L \simeq k_\mu / M_A \sim \mathcal{O}(E/M_A) \gg 1$, i.e., enhanced vs. ϵ_μ^T being $\mathcal{O}(1)$ always

- we can deduce sum over polarization vectors ("completeness relation"):

$$\boxed{\sum_{\substack{r=1,2,3 \\ (\text{or } 2T,\, 1L)}} \epsilon_{r\mu}(k)\, \epsilon^*_{r\nu}(k) = \left(-g_{\mu\nu} + \frac{k_{\mu\nu}}{M^2}\right)}$$

- Indeed, above sum over 3 physical polarizations "matches" form of massive gauge boson propagator obtained in unitarity gauge, i.e.,

$$\frac{1}{k^2 - M_A^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_A^2}\right),$$

which turns out to be same as for Proca Lagrangian (with explicit mass term: recall we get well-defined propagator, since no gauge invariance due to mass term, cf. massless gauge field): after all, quadratic terms in A_μ are same in Proca & SSB Lagrangians

— This confirms that unitarity gauge is the physical gauge, i.e., it does not contain any "unphysical" d.o.f. (cf. general / R_ξ gauge in next note): again, in bosonic sector, these are η (massive scalar) & 3 polarizations of (massive) A_μ

— However, even if physical content theory is manifest in unitarity gauge, calculation of loops (especially divergences therein, thus the issue of renormalizability) is tricky, e.g., it's not clear if we "remain" in unitary gauge upon quantum corrections...

— Actually, as mentioned above gauge boson propagator in unitarity gauge (obtained from SSB) is identical to that with explicit mass term, which was argued

to possibly have "problems" with renormalizability
... SSB/Higgs mechanism as "replacement" for explicit mass term was motivated for "improving" on renormalizability front

- But, $\boxed{\text{naively}}$ it looks like we might be back-to-square one then ?!

... maybe/hopefully, not quite, since are 2 theories really the same ?

- No, since SSB model has additional scalar, η (Higgs boson), which is absent with explicit mass term ... i.e., a "remnant" of gauge symmetry (sign that it's SSB or gentle breaking) which could save the day (a la relations between non-renormalizable couplings in radial representation for global, continuous SSB controlling divergences) ?!

- In order to "test" whether this direction/hope is worth pursuing, let us try to see if η (Higgs boson) can be made ∞ heavy (decoupled), while keeping gauge/fermion masses & fermion-gauge coupling fixed ... so that SSB model really reduces to explicit mass term

– Indeed, with fermion coupled vectorially to A_μ as assumed above, we see that this is doable: take $Q_\phi \to 0$, $v \to \infty$ (with e fixed) such that $M_A = e Q_\phi v$ is kept constant (of course, m_ψ untouched, so is fermion-gauge boson coupling).

So, for fixed (perturbative) λ, we see that $m_\eta^2 = 2\lambda v^2 \to \infty$ as desired: again, suitable limit of SSB/Higgs model is identical to explicit mass term for gauge boson, to which fermion couples vectorially!

— However, we had argued earlier, i.e., with explicit mass term, that this model is renormalizable (since "bad" part of propagator does not contribute to amplitudes)... so, in this sense, there was "no need" to go to SSB/Higgs model (we will of course show in next note that Higgs mechanism is renormalizable): a nice consistency check

— what about the other case for fermion-gauge boson coupling that we discussed in context of explicit gauge boson mass term? Namely, an axial-vector part to this coupling, e.g., suppose only L chirality of ψ couples to A_μ (with charge 1), i.e., R chirality has zero charge:

$$\mathcal{L} \ni i\bar{\Psi}_L \underbrace{\not{D}}_{\partial_\mu + ie \cdot \underset{\uparrow \, Q_\psi}{1} A_\mu} \psi_L + i\bar{\Psi}_R \not{\partial} \psi_R$$

no A_μ here

$$\ni -e A_\mu \left(j^\mu_V - j^\mu_A \right)$$ [i.e., (V−A) coupling]

– Back then, we argued that this theory could be non-renormalizable, since "bad" part of propagator contributes $\propto \partial_\mu j^\mu_A \propto m_\psi$ (bare mass term) $\neq 0$

$\Rightarrow$ in this axial-coupling case, we do need SSB/Higgs model for saving renormalizability

$\Rightarrow$ It better **not** be the case that by decoupling η, SSB/Higgs model can be reduced to explicit mass term model (again, which is non-renormalizable)... otherwise this was waste of effort!

– Indeed, in SSB/Higgs model, we can**not** quite play the earlier game of decoupling η for axial case.

- Namely, a bare mass term for fermion is not gauge-invariant (note that $\mathcal{L}$ is fully gauge-invariant)
- Instead, m_ψ (like gauge boson mass) must be generated by SSB, i.e., coupling ψ to Φ ... which requires $Q_\phi = 1$:

$$\mathcal{L} \ni h \underbrace{\overline{\psi}_L}_{-1} \underbrace{\Phi}_{+1} \underbrace{\psi_R}_{0} + h.c. \quad \Rightarrow \text{total} = 0$$

charges: $-1 \quad +1 \quad 0$

(we can easily show such a term is gauge-invariant.)

with $\boxed{m_\psi = h\, v/\sqrt{2}}$ ...

... thus, we cannot take $Q_\phi \to 0$, $v \to \infty$ ... unlike for vector case

[Equivalently, we could decouple

η like before using $Q_\phi \to 0 \ldots$
but then we can not couple Φ to
ψ in gauge-invariant way $\Rightarrow$
$m_\psi \to 0$, i.e., we can reduce SSB/Higgs
model (again, to be shown is renormalizable)
to explicit gauge boson mass term,
but only if ψ (coupling with axial
part) is massless ...
... but in that limit, we already (i.e., with
explicit gauge boson mass term) knew
theory is renormalizable, since
$\partial_\mu j^\mu_A \propto m_\psi \to 0$ so that "bad"
part of propagator does not contribute

... again, a sanity check: SSB/Higgs
model has a "chance" to be renormalizable,
even for axial coupling, although the
(different) model with explicit gauge
boson mass term in non-renormalizable

– How do we make sure above hope is actually realized ?! We need to "clarify" renormalizability feature ...

... recall the advantage of choosing representations in this regard, e.g., linear representation for scalar in global, continuous SSB model was "obviously" renormalizable, while this was "hidden" in radial representation (on the other hand, derivative nature of NGB interactions was not manifest in linear representation)

$\Rightarrow$ go back to linear representation for scalar in Higgs model, resulting in general / R_ξ-gauge, which makes renormalizability transparent ...

... but "price" to pay in R_ξ-gauge is that it involves unphysical d.o.f. [e.g., we have to "resurrect" NGB, $\xi(x)$ and time-like/scalar polarization of gauge boson, but only as internal lines (after all, this is not unitarity gauge!) ...with these 2 unphysical effects "canceling"]

— Rough analogy with quantization of (massless) photon field in QED: if we only use two physical (i.e., transverse) polarizations of photon, Lorentz invariance is not manifest (but of course it's present nonetheless) ... while adding two unphysical polarizations (but not as external lines), i.e., longitudinal and time-like/scalar, makes it easier to keep track of Lorentz invariance (contributions of these 2 unphysical polarizations "cancel" in net amplitude (like for Higgs model above)