19

Spontaneously Broken Global Symmetries

Much of the physics of this century has been built on principles of symmetry: first the spacetime symmetries of Einstein's 1905 special theory of relativity, and then internal symmetries, such as the approximate $SU(2)$ isospin symmetry of the 1930s. It was therefore exciting when in the 1960s it was discovered that there are more internal symmetries than could be guessed by inspection of the spectrum of elementary particles. There are exact or approximate symmetries of the underlying theory that are 'spontaneously broken,' in the sense that they are not realized as symmetry transformations of the physical states of the theory, and in particular do not leave the vacuum state invariant. The breakthrough was the discovery of a broken approximate global $SU(2) \times SU(2)$ symmetry of the strong interactions, which will be discussed in detail in Section 19.3. This was soon followed by the discovery of an exact but spontaneously broken local $SU(2) \times U(1)$ symmetry of the weak and electromagnetic interactions, which will be taken up along with more general broken local symmetries in Chapter 21. In this chapter we shall begin with a general discussion of broken global symmetries, and then move on to physical examples.

19.1 Degenerate Vacua

We do not have to look far for examples of spontaneous symmetry breaking. Consider a chair. The equations governing the atoms of the chair are rotationally symmetric, but a solution of these equations, the actual chair, has a definite orientation in space. Here we will be concerned not so much with the breaking of symmetries by *objects* like chairs, but rather with the symmetry breaking in the ground state of any realistic quantum field theory, the vacuum.

A spontaneously broken symmetry in field theory is always associated with a degeneracy of vacuum states. For instance, consider a symmetry transformation of the action, and of the measure used in integrating over fields, that acts linearly on a set of scalar fields $\phi_n(x)$:

$$\phi_n(x) \to \phi'_n(x) = \sum_m L_{nm}\phi_m(x) . \tag{19.1.1}$$

(The ϕ_n need not be elementary fields; they can be composite objects, like $\bar{\psi}\Gamma_n\psi$.) As we saw in Section 16.4, the quantum effective action $\Gamma[\phi]$ will then have the same symmetry

$$\Gamma[\phi] = \Gamma[L\phi] . \tag{19.1.2}$$

For the vacuum the expectation value of $\phi(x)$ must be at a minimum of the vacuum energy $-\Gamma[\phi]$, say at $\phi(x) = \bar{\phi}$ (a constant). But if $L\bar{\phi} \neq \bar{\phi}$, then this vacuum is not unique; $-\Gamma[\phi]$ has the same value at $\phi = L\bar{\phi}$ as it does at $\bar{\phi}$. In the simple special case where the symmetry transformation (19.1.1) is a reflection, $\phi \to -\phi$, if $-\Gamma(\phi)$ has a minimum at a non-zero value $\bar{\phi}$ of ϕ, then it has two minima, at $\bar{\phi}$ and $-\bar{\phi}$, each corresponding to a state of broken symmetry.

We are not yet ready to conclude that in such cases the symmetry is broken, because we have not yet ruled out the possibility that the true vacuum is a linear superposition of vacuum states in which ϕ_m has various expectation values, which would respect the assumed symmetry. For instance, in a theory with a symmetry $\phi \to -\phi$, even if $\Gamma(\phi)$ has a minimum for some non-zero value $\bar{\phi}$ of ϕ, how do we know that the true vacuum is one of the states $|\text{VAC}, \pm\rangle$ for which Φ has expectation values $\bar{\phi}$ and $-\bar{\phi}$, and not some linear combination like $|\text{VAC}, +\rangle + |\text{VAC}, -\rangle$ that *would* respect the symmetry under $\phi \to -\phi$? The assumed symmetry under the transformation $\phi \to -\phi$ tells us that the vacuum matrix elements of the Hamiltonian are

$$\langle \text{VAC}, +|H|\text{VAC}, +\rangle = \langle \text{VAC}, -|H|\text{VAC}, -\rangle \equiv a$$

(with a real) and

$$\langle \text{VAC}, +|H|\text{VAC}, -\rangle = \langle \text{VAC}, -|H|\text{VAC}, +\rangle \equiv b ,$$

(with b real), so the eigenstates of the Hamiltonian are $|\text{VAC}, +\rangle \pm |\text{VAC}, -\rangle$, with energies $a \pm |b|$. These energy eigenstates are invariant (or invariant up to a sign) under the symmetry $\phi \to -\phi$. In fact, the same issue also arises for chairs. The quantum mechanical ground state of an isolated chair is actually rotationally invariant; it is a state with zero angular momentum quantum numbers, and hence with no definite orientation in space.

Spontaneous symmetry breaking actually occurs only for idealized systems that are infinitely large. The appearance of broken symmetry for a chair arises because it has a macroscopic moment of inertia I, so that its ground state is part of a tower of rotationally excited states whose energies are separated by only tiny amounts, of order $\hbar^2/I$. This gives the state vector of the chair an exquisite sensitivity to external perturbations; even very weak external fields will shift the energy by much more than the energy difference of these rotational levels. In consequence, any rotationally asymmetric external field will cause the ground state or any other state of the chair with definite angular momentum numbers rapidly to develop components with other angular momentum quantum numbers. The states of the chair that are relatively stable with respect to small external perturbations are not those with definite angular momentum quantum numbers, but rather those with a definite orientation, in which the rotational symmetry of the underlying theory is broken.

For the vacuum also, the possibility of spontaneous symmetry breaking is again related to the large size of the system, specifically to the large volume of space. In the above example of a reflection symmetry, the off-diagonal matrix element b of the Hamiltonian involves an integration over field configurations that tunnel from the minimum at $\phi = \bar{\phi}$ to the one at $\phi = -\bar{\phi}$, so it is smaller than the diagonal matrix element a by a barrier penetration factor that for a spatial volume $\mathscr{V}$ is of the form $\exp(-C\mathscr{V})$, where C is a positive constant* depending on the microscopic parameters of the theory. The two energy eigenstates $|\text{VAC},+\rangle \pm |\text{VAC},-\rangle$ are thus essentially degenerate for any macroscopic volume, and so are strongly mixed by any perturbation that is an odd functional of ϕ. Even if such a perturbation H' is very weak, its diagonal elements $\langle \text{VAC},\pm|H'|\text{VAC},\pm\rangle$ will differ by much more than the exponentially suppressed off-diagonal elements of either H or the perturbation. Thus the vacuum eigenstates of the perturbed Hamiltonian will be very close to either one of the broken symmetry states $|\text{VAC},\pm\rangle$ which diagonalize the perturbation, and *not* to the invariant states $|\text{VAC},+\rangle \pm |\text{VAC},-\rangle$. Which one of the states $|\text{VAC},\pm\rangle$ is the true vacuum for very small perturbations? This depends on the perturbation, but since these two states are related by a symmetry transformation of the original Hamiltonian, it doesn't matter; if the perturbation is sufficiently small, no observer will be able to tell the difference.

The vanishing of matrix elements between vacuum states with different field expectation values becomes exact in a space of infinite volume.[1] For infinite volume, a general vacuum state $|v\rangle$ may be defined as a state with

* For instance, by analogy with the classic wave mechanical problem of barrier penetration, for a Lagrangian density of the form $-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, we have $C = \int_{-\bar{\phi}}^{+\bar{\phi}} \sqrt{2V(\phi)}\,d\phi$. We will not bother to calculate the off-diagonal matrix element b here, because we shall soon give a general argument that shows that it vanishes for infinite volume.

zero momentum

$$\mathbf{P}|v\rangle = 0 \tag{19.1.3}$$

for which this is a *discrete* momentum eigenvalue. (This excludes single-particle or multiparticle states, for which the momentum value zero is always part of a continuum of momentum values in a space of infinite volume.) In general there may be a number of such states. They can usually be expanded in a discrete set, and our notation will treat them as if they were discrete. They will be chosen to be orthonormal

$$\langle u|v\rangle = \delta_{uv} \,. \tag{19.1.4}$$

Any matrix element of a product of local Hermitian operators at equal times between these states may be expressed as a sum over states:

$$\langle u|A(\mathbf{x})\,B(0)|v\rangle = \sum_w \langle u|A(0)|w\rangle\;\langle w|B(0)|v\rangle + \int d^3p \sum_N \langle u|A(0)|N,\mathbf{p}\rangle\;\langle N,\mathbf{p}|B(0)|v\rangle e^{-i\mathbf{p}\cdot\mathbf{x}} \,, \tag{19.1.5}$$

where $|N,\mathbf{p}\rangle$ are a set of orthonormalized continuum states of definite three-momentum $\mathbf{p}$ that together with the $|v\rangle$ span the whole physical Hilbert space. (Here N may include continuous as well as discrete labels. Also, we are dropping time arguments.) We assume without proof that because the $|N,\mathbf{p}\rangle$ belong to the continuous spectrum of the momentum operator $\mathbf{P}$, the dependence of matrix elements on $\mathbf{p}$ is smooth enough (that is, Lebesgue integrable) to allow the use of the Riemann–Lebesgue theorem,[2] so that the integral over $\mathbf{p}$ vanishes as $|\mathbf{x}| \to \infty$. In this limit, we have then

$$\langle u|A(\mathbf{x})\,B(0)|v\rangle \xrightarrow[|\mathbf{x}|\to\infty]{} \sum_w \langle u|A(0)|w\rangle\;\langle w|B(0)|v\rangle \,. \tag{19.1.6}$$

Likewise,

$$\langle u|B(0)\,A(\mathbf{x})|v\rangle \xrightarrow[|\mathbf{x}|\to\infty]{} \sum_w \langle u|B(0)|w\rangle\;\langle w|A(0)|v\rangle \,. \tag{19.1.7}$$

But causality tells us that the equal-time commutator $[A(\mathbf{x}), B(0)]$ vanishes for $\mathbf{x} \neq 0$ (see Section 5.1), so the matrix elements (19.1.6) and (19.1.7) are equal, and thus the Hermitian matrices $\langle u|A(0)|v\rangle$, $\langle u|B(0)|v\rangle$, etc., must all commute with one another. It follows that they can all be simultaneously diagonalized. Changing if necessary to this basis, we have then for every Hermitian local operator $A(\mathbf{x})$ of the theory

$$\langle u|A(0)|v\rangle = \delta_{uv}\, a_v \tag{19.1.8}$$

with a_v a real number, the expectation value of A in the state $|v\rangle$. So for infinite volume any Hamiltonian constructed from local operators will have vanishing matrix elements between the different vacua $|v\rangle$. In the absence of off-diagonal terms in the Hamiltonian, any two $|v\rangle$s connected by a symmetry operation will be degenerate. A symmetry-breaking perturbation built out of such local operators will be diagonal in the same basis, and will therefore yield a ground state that is one of the $|v\rangle$s, rather than a linear combination of them.

It is reassuring that the vacuum states $|v\rangle$ which are stable against small field-dependent perturbations are also vacuum states in which the cluster decomposition condition (see Chapter 4) is satisfied. This principle requires that for the physical vacuum state $|\mathrm{VAC}\rangle$

$$\langle \mathrm{VAC}|A(\mathbf{x})B(0)|\mathrm{VAC}\rangle \xrightarrow[\mathbf{x}\to\infty]{} \langle \mathrm{VAC}|A(\mathbf{x})|\mathrm{VAC}\rangle\, \langle \mathrm{VAC}|B(0)|\mathrm{VAC}\rangle\,. \quad (19.1.9)$$

This condition is satisfied if we take the vacuum state $|\mathrm{VAC}\rangle$ to be any one of the states $|v\rangle$ in the basis defined by Eq. (19.1.8), but not if we take it to be a general linear combination of several of the $|v\rangle$s.

19.2 Goldstone Bosons

We now specialize to the case of a spontaneously broken *continuous* symmetry. In this case there is a theorem, that (with one important exception, to be considered in Chapter 21) the spectrum of physical particles must contain one particle of zero mass and spin for each broken symmetry. Such particles, known as Goldstone bosons (or Nambu–Goldstone bosons) were first encountered in specific models by Goldstone[3] and Nambu[4]; two general proofs of their existence were then given by Goldstone, Salam, and myself.[5] This section will present both of these proofs, and then go on to consider the properties of the Goldstone bosons.

Suppose that the action and measure are invariant under a continuous symmetry, under which a set of Hermitian scalar fields $\phi_n(x)$ (either elementary or composite) are subjected to the linear infinitesimal transformations

$$\phi_n(x) \to \phi_n(x) + i\epsilon \sum_m t_{nm}\phi_m(x)\,, \quad (19.2.1)$$

with it_{nm} a finite real matrix. As we found in Section 16.4, the effective action is then also invariant under this transformation

$$\sum_{n,m} \int \frac{\delta\Gamma[\phi]}{\delta\phi_n(x)}\, t_{nm}\phi_m(x)\, d^4x = 0\,. \quad (19.2.2)$$

We shall specialize to the case of a translationally invariant theory with constant fields ϕ_n, where as we saw in Section 16.2, the effective action