Comments on pion (π)-nucleon (N) system

capital ψ (not $\bar{\psi}$!)

(1). $\boxed{SU(2)}$ acting on fermion doublet $\Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$:

$\leftarrow$ Pauli matrices

$$\boxed{\Psi \rightarrow \exp\left(-i \beta_a \sigma^a/_2\right) \Psi}$$

scalar triplet

is isomorphic to $\boxed{SO(3)}$ acting on $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$:

$$\boxed{\Phi \rightarrow \exp\left(-i \beta_a T_a^{adj.}\right) \Phi}$$

where $\left(T_a^{adj.}\right)_{bc} = -i \epsilon_{abc}$ $(\epsilon_{123} = -\epsilon_{132} = 1 \dots)$

since Lie algebra/commutation relations are the same:

$$\boxed{\left[T^a, T^b\right] = i \epsilon^{abc} T_c} \quad \text{for } T_a = \frac{\sigma_a}{2} \text{ or } T_a^{adj.}$$

$\frac{\sigma_a}{2}$: 2×2 matrices; $T_a^{adj.}$: 3×3 matrices

$\Longrightarrow$ elements of group generated by $T_a^{adj.}$ (again 3×3 matrices) are in one-to-one correspondence with those generated by $\sigma_a/_2$ (2×2 matrices). Namely, if we have

$$\underbrace{\exp\left(-i \beta_a^{(1)} \sigma_a/_2\right) \times \exp\left(-i \beta_b^{(2)} \frac{\sigma_b}{2}\right)}_{\substack{\text{product of 2 group elements:} \\ (2\times 2) \text{ unitary matrices}}} = \underbrace{\exp\left(-i \beta_c^{(3)} \frac{\sigma_c}{2}\right)}_{\substack{\text{another group} \\ \text{element}}},$$

then (necessarily),

$$\underbrace{\exp\left[-i \beta_a^{(1)} T_a^{adj.}\right]}_{\substack{(3\times 3) \text{ orthogonal} \\ \text{matrix}}} \times \exp\left[-i \beta_b^{(1)} T_b^{adj}\right] = \exp\left[-i \beta_c^{(1)} T_c^{adj}\right]$$

(2). Summary of symmetries : we have

$\boxed{\text{field} \longrightarrow \exp\left[-i\beta_a T^a - i\alpha B\right] \times \text{field}}$, i.e.,

$\boxed{SU(2) \times U(1)_B}$, where $B = \mathbb{1}_{2\times 2}$ for fermion doublet ; $B = 0$ for scalar triplet ;

$T^a_{ij} = (\sigma^a/2)_{ij}$ $(a = 1,2,3 \, ; \, i,j = 1,2)$ for fermion doublet and $(T^a)_{bc} = \big(\underbrace{T_a^{\text{adj.}}}_{\text{adjoint representation}}\big)_{bc} = -i\epsilon_{abc}$ for scalar triplet

— $\boxed{U(1)_{EM}}$ [local/gauge symmetry, after adding photon] is a subgroup of (above) $SU(2) \times U(1)_B$ generated by $\boxed{Q_{EM} = \underbrace{T_3}_{\text{3rd component of } SU(2) \text{ [isospin]}} + \frac{1}{2} B}$, i.e.,

$\boxed{\text{field} \longrightarrow \exp\left[-i\beta_3 \underbrace{T^3}_{\sigma^3/2 \text{ for fermion or } -i\epsilon_{3bc} \text{ for scalar}} - i\beta_{3/2} B\right] \times \text{field}}$

[set $\beta_1 = \beta_2 = 0$ and $\frac{\beta_3}{2} = \alpha$ in the full symmetry transformation]