Non-abelian gauge theories (quantum) ①

— Bottomline (but takes a while to get there!): contribution to vacuum polarization (gauge boson self-energy) from gauge loops [in turn, due to self-interactions of gauge bosons already seen in classical case from $\mathcal{L}_{\text{pure gauge}}$] tend to make theory asymptotically free, i.e., gauge coupling becomes smaller in the UV (higher energies): this is the $\boxed{\text{new}}$ feature of non-abelian gauge theories. Matter contributions (at 1-loop level) are IR-free (just like for abelian case)

— $\boxed{\text{Outline}}$ of quantization procedure: canonical approach is cumbersome here for 2 reasons, i.e.,

(1) interactions involve derivatives necessarily (again, in gauge boson self-couplings: dictated by gauge invariance). [Note that such couplings can arise even in abelian gauge theories, e.g., gauge couplings of a scalar field, but in a sense they were "optional"!]

— This issue is relatively easy to spot, and its solution is also "intuitive" (as we will see)

(2) special for non-abelian case is "difficulty" in maintaining gauge invariance when computing

diagrams involving loops of gauge bosons (again, coming from self-interactions).

— However, this problem is rather unexpected from get-go (cf. one above) and its solution is also not so straightforward (see more details below)

— So, Feynman's $\boxed{\text{path/functional integral}}$ is better suited for quantizing non-abelian gauge theories. However, studying this is sort of beyond scope of this course (it's possible that Phys 851 might cover it)

— So, here we will follow the approach of simply presenting the Feynman rules resulting from path integral (PI) formalism, making them "plausible" using canonical approach (it turns out that some — but not all — such educated guesswork is correct in the/end, i.e., justified rigorously / a posteriori by PI approach)

— In particular, we will encounter $\boxed{2}$ sets of unphysical states in this process, i.e.,

(a) scalar/time-like and longitudinal polarizations of gauge bosons: these are "needed" for maintaining manifest $\boxed{\text{Lorentz}}$ invariance (LI), just like in the case of QED, i.e., abelian gauge theory

"like" negative norm for unphysical polarizations

(b) "ghost" states (with "wrong" spin vs. statistics) maintain gauge invariance (GI): this is really a feature of non-abelian gauge theory (it is related to problem (2) mentioned above with canonical approach) ③

- Both these sets of states cannot appear as external lines, but again, enter as virtual states/internal lines, helping to keep LI & GI, respectively
- With above introduction/words, let's dig into details/formulae
- Recall that

$$\mathcal{L}_{\text{pure gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a, \text{ with } F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$$

contains a "$(\partial A) A^2$" term, i.e., derivative interactions

- Why is this a "problem" for canonical quantization? Note that in perturbation theory, we first quantize the "free" part (i.e., quadratic in fields) of Lagrangian, treating rest of terms as interactions
- So, assuming there are $\boxed{\text{no}}$ derivatives in interactions part, then the canonical/conjugate momentum, $\pi = \partial(\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int.}}) / \partial \dot{\phi}$ is $\boxed{\text{not}}$ modified (vs. free case) by interactions [i.e., equal-time (anti-)commutation relations are same as free case]

$$\Rightarrow \boxed{\text{Hamiltonian}},\ \mathcal{H} = \underbrace{\pi}_{\text{(again) } \partial \mathcal{L}_{\boxed{\text{free}}} / \partial \dot{\phi}} \dot{\phi} - (\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}})$$

$$\equiv \mathcal{H}_{\text{free}} + \underbrace{\mathcal{H}_{\text{int}}}_{= \boxed{-\mathcal{L}_{\text{int}}}}$$ in this case (that's crucial here)

- Recall that $\mathcal{H}_{\text{int}}$ ($= -\mathcal{L}_{\text{int}}$ here) determines $\boxed{\text{S-matrix}}$ elements

- However, if instead $\mathcal{L}_{int.}$ contains derivatives (as in the case of non-abelian gauge theory), then (at least naively) $\boxed{\Pi}$ is modified relative to Π_{free} (due to $\partial \mathcal{L}_{\boxed{int}} / \partial \dot{\phi}$ contribution). Then $\mathcal{H}_{int} \neq -\mathcal{L}_{int}$ $\boxed{?!}$
- It turns out (as seen from $\boxed{PI}$ approach) that even in this case, we actually (effectively) have $\boxed{\mathcal{H}_{int} = -\mathcal{L}_{int}}$ (again, this is valid for other theories with derivative couplings, e.g., scalar QED) $\Rightarrow$ Feynman rule for derivative interaction is obtained "as usual" from $\mathcal{L}_{int}$, i.e., $\boxed{\partial_\mu - p_\mu}$ (momentum) etc. [as we mentioned for NGB couplings in radial representation in HW 3.4]
- So, issue (1) mentioned in outline (for non-abelian case) is taken care of, i.e., free $\mathcal{L}$ is quadratic part of $\mathcal{L}_{pure\ gauge}$ (with tri-linear & quartic pieces being interactions) and this part is same as $\mathcal{L}_{free}$ for $\boxed{U(1)}$, apart from index "a" on A_μ ...

... which leads to another "problem/issue" in (gauge field) quantization, well-known from photon/QED (so it wasn't really listed in (above) outline). Namely, $\boxed{A_\mu}$ (for each "a") has only $\boxed{2}$ physical polarizations (transverse), but quantizing only these results in loss of manifest

LI, i.e., instantaneous Coulomb-type interaction (not from exchange of photons) has to be then treated as classical potential (see, for example, Mandl, Shaw page 81) ⑤

- So, it's more convenient (from LI viewpoint) to treat $\boxed{\text{all}}$ 4 components/polarizations of gauge field similarly (i.e., quantize all 4), but then "gauge-fix", i.e., impose subsidiary condition (or suitable constraint) in order to handle this "redundancy" in d.o.f.
- In particular, the extra, unphysical polarizations (scalar/time-like & longitudinal) are dynamical, but only propagate as internal/virtual lines in Feynman diagrams, i.e, are "forced" to not be external lines (again, we do this also in QED)
- So with addition of $\mathcal{L}_{\boxed{\text{gauge-fixing}}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu_a\right)^2$

  "new" for non-abelian

  we get $\boxed{\text{propagator}}$ (assigned to internal line)

$$\boxed{D^{ab}_{\mu\nu}(k) = -\frac{\overset{\text{new}}{\delta_{ab}}}{k^2+i\epsilon}\left[g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}$$

$\mu, a \xrightarrow{k} \nu, b$

- Onto $\boxed{\text{Feynman rules}}$ for self-coupling of gauge bosons

We have $\mathcal{L}_{\text{pure gauge}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$

$$= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc}A^b_\mu A^c_\nu\right)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{ade}A^d_\mu A^e_\nu\right)$$

$\longrightarrow$ $\boxed{\text{different}}$ dummy indices

$= -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^{\text{"}2\text{"}}$ (free part, giving above propagator, after gauge-fixing) 6

$+ \frac{1}{2} g f_{abc} A_b^\mu A_c^\nu \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)$ ... (1) (cubic/tri-linear interactions)

$- \frac{1}{4} g^2 f_{abc} f_{ade} A_\mu^b A_\nu^c A_d^\mu A_e^\nu$ (quartic coupling)

Note: there are no A_a^3 or A_a^4 (@ fixed) interactions due to antisymmetry of f_{abc}, f_{ade} [in this sense, it is kind of like photon in QED, ie, gauge boson of given index (a) doesn't interact with itself].

— Let's see how to derive $\boxed{\text{Feynman rule}}$ for cubic/tri-linear vertex (HW $\boxed{7.3}$ deals with quartic coupling)

— For this purpose, rewrite 2nd term on RHS of Eq. 1 as

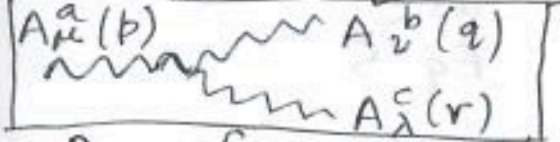
$$\mathcal{L}_{\text{pure gauge, cubic}} = g \overbrace{f_{a'b'c'} A_{b'}^\alpha A_{c'}^\beta \left(\partial_\alpha A_\beta^{a'}\right)}^{\text{note single term}} \quad \dots (2)$$

no "1/2" using antisymmetry of f's

— Suppose we want $\boxed{\text{Feynman rule}}$ for interaction of $A_\mu^a(p)$; $A_\nu^b(q)$ and $A_\lambda^c(r)$, where p, q, r are respective momenta going into vertex, i.e., gauge bosons with these momenta are being annihilated

— Claim: apart from polarization vectors (ϵ_μ etc.) for external lines [or $D_{\mu ..}^{a ..}$ for internal line], we assign to this vertex:

$$\boxed{-g f_{abc}\left[(q-r)_\mu g_{\nu\lambda} + (r-p)_\nu g_{\lambda\mu} + (p-q)_\lambda g_{\mu\nu}\right]} \quad \ldots(3)$$



Proof: we show that $\boxed{2^{nd}}$ term in Eq. 3 originates from using $(A^\alpha_{b'})$ field in Eq. 2 ($\mathcal{L}_{cubic}$) in order to annihilate $\boxed{A^b_\nu(q)}$ gauge boson/particle [other 2 terms in Eq. 3 follow from remaining 2 possibilities for annihilating $A^b_\nu(q)$, i.e., using instead $A^\beta_{c'}$ or $A^{a'}_\beta$ in Eq. 2]

- Clearly, we then match $\underline{b' = b}$ & $\underline{\alpha = \nu}$ ...(4)

of gauge boson/particle

(again) of field in Eq. 2

- We then have 2 ways to annihilate $\boxed{A^a_\mu(p)}$: ...(5)

$\boxed{(i)}$ use $\boxed{A^\beta_{c'}}$ in Eq. 2 so we set $\underline{c' = a}$ & $\underline{\beta = \mu}$

- It follows that $\boxed{A^c_\lambda}$ (3rd gauge boson/particle) is then taken care of by $(\partial_\alpha A^{a'}_\beta)$ in Eq. 2, ...(6) giving $\underline{a' = c}$ & $\underline{\beta = \lambda}$ and a momentum factor of $\boxed{-i r_\alpha} = -i r_\nu$ (using Eq. 4): recall that annihilation operator in Fourier expansion of gauge field has $e^{-ir.x}$ factor, upon which $\boxed{\partial_\alpha}$ acts

- So, net factor for piece (i) is:

$(-i r_\nu)(i)\, g\, f_{(a'=c)(b'=b)(c'=a)}$

from (i) $\mathcal{L}_{int}$... use Eq. 6, use Eq. 4, use Eq. 5

$\times\, g_{\lambda\mu}$ ← use $\beta=\mu$ (Eq. 5) and $\beta=\lambda$ (Eq. 6)

i.e., we get $+ g_{\lambda\mu} r_\nu f_{cba} = \boxed{(-) g_{\lambda\mu} r_\nu f_{abc}}$ ...(7)

use antisymmetry of f

$\boxed{\text{(ii)}}$ we can instead use $\boxed{\partial_\alpha A_\beta^{a'}}$ in Eq. 2 in order to annihilate $\boxed{A_\mu^a(p)}$, giving $a'=a$ & $\beta=\mu$ and $\boxed{-i p_{\alpha=\nu}}$ (again, from $\boxed{\partial_\alpha}$ acting on $e^{-ip\cdot x}$)

– And, use $\boxed{A^\beta_{c'}}$ in Eq. 2 to annihilate $\boxed{A_\lambda^c}$, leading to $c'=c$ & $\beta=\lambda$

so that net factor is (skipping some details)

$(-i p_\nu)(i)\, g\, f_{(a'=a)(b'=b)(c'=c)} \times g_{\lambda\mu}$ ($\beta=\lambda$ & $\beta=\mu$)

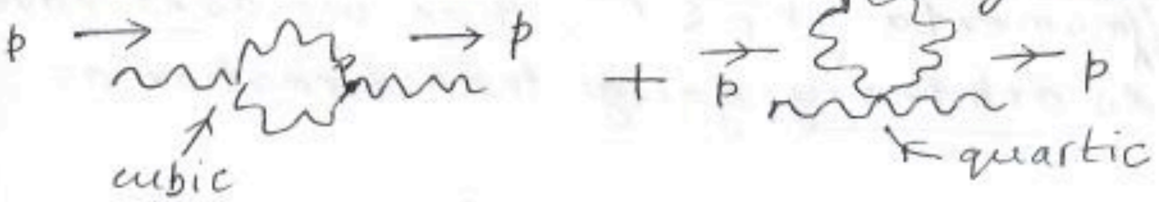
$= \boxed{p_\nu g_{\lambda\mu} f_{abc}}$ ...(8)

– Combining Eqs. 7, 8 gives

$\boxed{-g_{\lambda\mu} f_{abc}(r-p)_\nu}$ as in 2nd term of Eq. 3

[Of course, if momentum is instead outgoing, i.e., gauge boson is created, then ∂ acts on $e^{+ip\cdot x}$. So, $\boxed{p\to(-)p}$ in rule above

— $\boxed{\text{"Armed"}}$ with these Feynman rules (for propagators and vertices), we calculate vacuum polarization amplitude from gauge loops only:



... when we encounter (rather unexpectedly, cf. derivative interactions issue) $\boxed{\text{reason (2)}}$ why canonical quantization "doesn't work", i.e., we don't find the tensor structure "expected" from gauge invariance (generalization of Ward identity discussed for QED to non-abelian case):

$\boxed{(p^2 g_{\mu\nu} - p_\mu p_\nu)}$ ... even if we use a regulator for UV divergence which preserves gauge invariance, e.g. DIMREG

— Recall that in $\boxed{\text{QED}}$, we faced a similar situation, i.e., got the superficial quadratic divergence $[\Lambda^2 g_{\mu\nu}]$ in vacuum polarization amplitude if we used a (hard) cut-off (Λ) regularization

- However, the "diagnosis" for above QED case was simply that hard $\boxed{\text{cut-off}}$ does not preserve gauge invariance as follows. Recall that under gauge transformation, we have

$$\psi \rightarrow e^{-ieQ\theta(x)}\psi \quad \text{and} \quad A_\mu \rightarrow A_\mu + \partial_\mu \theta(x)$$

so if $\theta(x)$ contains Fourier (frequency) modes above Λ, then gauge transformation moves low ($<\Lambda$) modes in original Ψ, A_μ into modes above cut-off in new Ψ, A (which is not supposed to be "allowed"), i.e., if we insist on modes in Ψ, A_μ having only frequencies/Euclidean momenta, $k_E^2 \leq \Lambda^2$, then we don't have freedom to do arbitrary gauge transformation as above

— $\boxed{\text{Back}}$ to non-abelian vacuum polarization: above digression about hard cut-off usage in QED was just to highlight that we should "hunt" for an underlying "reason" for violation of gauge invariance (in turn, not giving right tensor structure) for non-abelian vacuum polarization.

— Indeed, in hindsight (which is always "20/20"!), the following intuitive argument/educated guess (thus not really "proof") might do it. Recall that $\mathcal{L}_{\boxed{\text{free}}}$ for non-abelian gauge fields (i.e., quadratic in A_μ^a), which is used to obtain spectrum of particles & propagators, is $\boxed{\text{not}}$ gauge-invariant by itself. Again, $(F_{\mu\nu}^a)^2 \sim \underbrace{(\partial A)^2}_{\text{free}} + \underbrace{\partial A A^2 + A^4}_{\text{cubic, quartic}}$ containing $\mathcal{L}_{\boxed{\text{free}}}$ is gauge-invariant, but we shove part of it into interactions (cubic, quartic in A_μ^a): we "have" to do such a "split" of $(F_{\mu\nu}^a)^2$ in order to do canonically quantize/do perturbation theory, i.e., we can't treat full (gauge-invariant) $(F_{\mu\nu}^a)^2$ as free/leading term

— In particular, one could then ask why $\boxed{\text{not}}$ include mass term for gauge field (A^2): it breaks gauge invariance, but so does $\mathcal{L}_{\boxed{\text{free}}}$ $\boxed{?!}$

— Contrast this non-abelian story with abelian case: $\boxed{F_{\mu\nu}^2}$ there $\boxed{\text{is}}$ gauge invariant and free [i.e., contains only $(\partial A)^2$]; we can forbid mass term (A^2) based on gauge invariance etc.

— So, inorder to maintain (manifest) gauge invariance in each step of perturbation theory, we "introduce" extra, unphysical (see below) fields called "Fadeev-Popov (FP) or ghost" fields, denoted by $\boxed{c_a(x)}$ and anti-ghost fields, $\boxed{b_a(x)}$ [where $a = 1 \ldots n^2-1$] (using Lahiri, Pal notation)

– These ghost fields "look like" scalar fields (in adjoint representation of non-abelian group), but in fact obey anti-commutation relations like fermions!

— So, clearly these ghost fields are unphysical (don't contradict usual spin-statistics in this sense), i.e., they cannot be external lines in Feynman diagrams, but it is "crucial" to include them as internal/virtual lines in order to keep gauge invariance throughout the calculation process [Note that a loop of ghost fields will give "−1" just like fermions]

[As an aside/reminder, inclusion of unphysical $\boxed{\text{virtual}}$ states/internal lines as a "trick/book keeping device", i.e., for maintaining a principle from start to finish, is by now a well-known strategy:

for example, scalar/time-like & longitudinal (unphysical/extra) photon (massless) polarizations are introduced in $\boxed{\text{QED}}$ in order to have manifest Lorentz invariance (recall scalar/time-like photon polarization has negative norm, akin to "wrong" spin-statistics for ghost fields here).

Similarly, for SSB of gauge theories / Higgs mechanism, i.e., for massive gauge boson, longitudinal polarization is of course also physical, but time-like/scalar is still unphysical: in general/$\boxed{R_\xi}$ gauge, we "keep" it, along with (would-be) NGB ($\mathcal{G}$), in order to make renormalizability transparent.

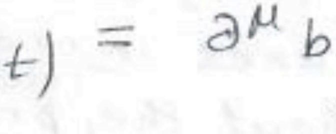
Or, at an even more basic level, in regularization procedures, we introduce unphysical parameters, e.g., Pauli-Villars field or going to $(4-2\epsilon)$ dimensions etc.]

— $\boxed{\text{Back}}$ to our business: we have

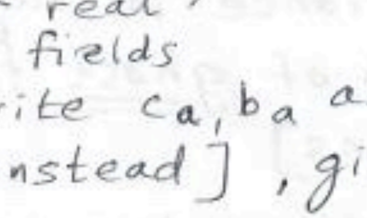
$$\mathcal{L}_{FP\,(\text{ghost})} = \partial^\mu b_a \left[\partial_\mu c_a + g f_{abc}\, c_b A^c_\mu\right]$$

real fields

[we can of course re-write c_a, b_a as (n^2-1) complex scalar fields instead], giving Feynman rules for $\boxed{\text{ghosts}}$:

propagator: 

$= i\delta_{ab}/k^2$

gaugeboson-(ghost)^2 vertex: 

$= g f_{abc}\, p^\mu$

so that we get the following (extra) contribution to vacuum polarization:

(−1) for "fermion" loop

— $\boxed{\text{Including}}$ above effect <u>does</u> result in (expected from gauge invariance) tensor structure, i.e., $(k^2 g_{\mu\nu} - k_\mu k_\nu)$, i.e., "restores" <u>gauge invariance</u>.

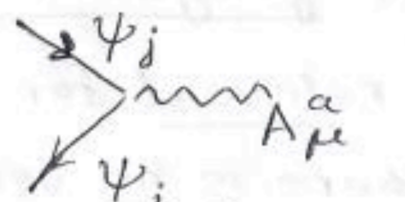
— clearly, ghost fields are <u>not</u> <u>relevant</u> for <u>tree-level</u> calculations (since there is <u>no</u> vertex with <u>one</u> ghost line only <u>and</u> they can't be external lines), $\boxed{\text{cf.}}$ inclusion of <u>scalar & longitudinal</u> (unphysical) <u>photon polarizations</u> in tree-level diagrams (as internal lines) is necessary in order to obtain <u>Coulomb</u> interaction between charged particles ... or, <u>tree</u>-level exchange of (would-be) <u>NGB</u> and <u>scalar</u> polarization of $\boxed{\text{massive}}$ gauge boson is needed in $\boxed{R_\xi}$ gauge etc.

— Also, <u>abelian</u> case can be obtained from above <u>non-abelian</u> by simply setting $\boxed{f_{abc} = 0}$, so we see that ghost fields in abelian theories do <u>not</u> <u>interact</u>, hence can be <u>neglected</u>.

— Finally, while introduction of ghost fields above seems somewhat "ad-hoc" (although "justified" by need to maintain gauge invariance), their "appearance" is quite "natural" in $\boxed{\text{PI}}$ <u>formalism</u>, which is how all of above is actually <u>derived</u>.

$\boxed{\text{Feynman rules}}$ for interactions of gauge bosons with $\boxed{\text{matter}}$ particles (simple generalization of QED, cf. gauge boson self-interactions above)

- From $\mathcal{L}_{int}$ for $\boxed{\text{fermions}}$ $= -g\, \bar{\Psi}_i \, (T^a)_{ij} \, \gamma^\mu A^a_\mu \, \Psi_j$, we get ← arbitrary representation



$$= -ig \overbrace{(T_a)_{ij}}^{\text{matrix in internal space (symmetry)}} \underbrace{\gamma_\mu}_{\text{matrix in Dirac/spinor space}}$$

Ψ_i → "component" of Ψ vector/column

- Basically, modifications relative to QED case are simply that

$\boxed{\text{(i)}}$ there is non-trivial matrix structure also in internal (symmetry) space, i.e., $(T^a)_{ij}$, so that gauge boson can couple off-diagonally (i.e., to two different $\boxed{\Psi\text{'s}}$), in addition to diagonally of course. Note that such off-diagonal coupling is needed to describe weak (nuclear) force, e.g. $\boxed{\bar{\mu}-\nu_\mu-W^+}$ vertex [this was another motivation to study non-abelian gauge theories].

$\boxed{\text{(ii)}}$ (Related to above) Coupling depends on $\boxed{\text{which}}$ of (n^2-1) gauge bosons are involved, i.e., $T^{a}_{..}$; again, some gauge bosons couple diagonally (like photon), while rest T^a's are $\boxed{\text{off}}$ diagonal

Similarly, $\mathcal{L}_{int.}$ for <u>scalar</u> fields (15)

comes from $(D_\mu \Phi)^\dagger (D^\mu \Phi)$, where $D_\mu \Phi = \partial_\mu \Phi + i g T_a A_\mu^a \Phi$,

i.e., (mixed) cubic & quartic terms :

$$i g (\partial_\mu \Phi^\dagger)(T_a A_a^\mu \Phi) - i g (T_a A_\mu^a \Phi)^\dagger (\partial_\mu \Phi)$$
$$+ g^2 (T_a A_\mu^a \Phi)^\dagger (T_b A_b^\mu \Phi)$$

like in QED, i.e., cubic/<u>tri-linear</u> ($2\ \phi$'s, $1\ A_\mu$) has <u>derivatives</u>, while <u>quartic</u> ($2\ \phi$'s, $2\ A_\mu$) does not.

(<u>HW</u> $\boxed{7.2}$ derives the corresponding <u>Feynman rules</u>.)