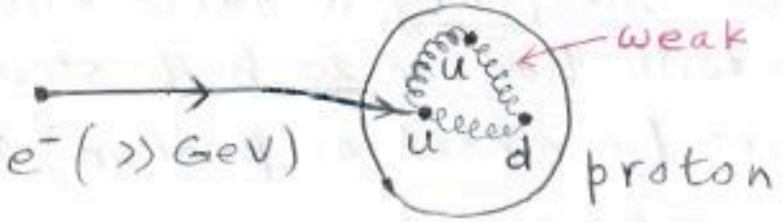
QCD phenomenology: details of 3 processes

(2). Deep inelastic scattering, i.e., scattering of leptons (electron or neutrino) off of hadrons (proton/neutron in simplest case) at high momentum transfer ($\gg$ GeV)

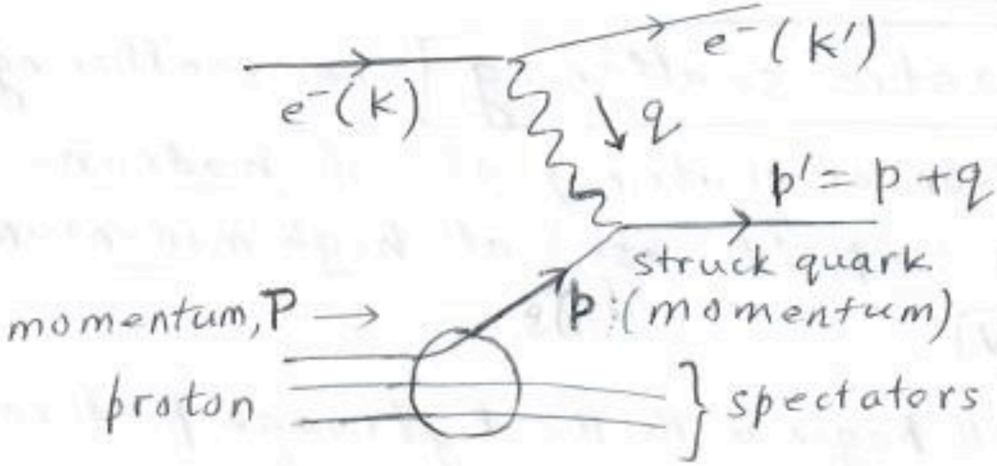
- As usual, we will begin with the big/rough picture, then slowly get into details: in the lab frame, (usually) hadron (say proton) is at rest, with which electron strikes with energy $\gg$ GeV, i.e., 1/size of proton.

At such high momentum transfer/ short distances (involved in underlying process), (like a "magnifying glass") electron "sees" proton sub-structure, i.e. quarks... inside it :



- Of course, the quarks inside proton interact with each other by gluon exchange, but due to asymptotic freedom, these couplings are weak at the high momentum transfer between electron and proton
- Hence, to leading order in α_s (@ $\gg$ GeV) $\ll 1$, electron-proton scattering (again, at high energy/momentum transfer) is pictured as electron scattering off of a single quark inside proton, with

remaining quarks (& gluons) inside struck proton being "spectators", i.e.,



— Clearly underlying process is then $\boxed{\text{EW}}$, e.g., photon/W_μ/Z_μ exchange ... which will get "corrected" (as outlined earlier) by $\boxed{\text{QCD}}$ effects in 3 ways [where 1st, 2nd are similar to those in process (1), i.e., $e^+e^- \to$ all hadrons], i.e., $\boxed{\text{(a).}}$ short distance/large energy transfer (>> GeV) – perturbative/calculable – will modify probability of electron quark scattering itself ; $\boxed{\text{(b).}}$ long distance effects/soft gluon exchange will hadronize both struck quark and spectators (remnant of proton) and $\boxed{\text{(c).}}$ probability to find quark of momentum p inside proton of momentum P is set also by long-distance (uncalculable) effects

— Let's write down some formulae now! We first note that (momentum transfer, $\boxed{q}$)$^2 = (k'-k)^2$

$$= \underbrace{k^2}_{m_e^2} + \underbrace{k'^2}_{m_e^2} - 2k\cdot k' \underset{\substack{\text{of } \boxed{\text{in}}\text{coming} \\ \text{electron, i.e.,} \\ k_\mu = (m_e, \bar{0})}}{\overset{\text{rest frame}}{=}} 2m_e^2 - 2m_e \underbrace{E'_e}_{\substack{\text{energy of} \\ \text{outgoing electron} \\ \geq m_e}} \boxed{<0}$$

- So, we define $\boxed{Q^2 = -q^2 \boxed{> 0}}$ which requires initial $\boxed{\text{electron}}$ energy in $\boxed{\text{lab}}$ frame ≫ GeV
- Suppose $\boxed{Q^2 \gg (\text{GeV})^2}$ (so that $\boxed{\text{ejected}}$ quark has energy/momentum $\sim\sqrt{Q^2}$ $\boxed{\gg \text{GeV}}$ in the lab frame (~p+q), since its initial energy/momentum p $\boxed{\sim \text{GeV}}$, i.e., proton mass/binding energy (again "p+q" ≫ GeV, while p~GeV)
- The initial, thus also final, energy/momentum of spectator quarks is $\boxed{\sim \text{GeV}}$ (again, in lab frame) (≲ GeV)
- Thus, roughly speaking, $\boxed{\text{soft}}$ gluon exchange (which underlies hadronization) betweem outgoing, struck quark and spectator/remnant quarks cannot "balance" the 2 involved momenta. [Of course, hard gluon exchange can do it, but these effects are suppressed by α_S (@ ≫ GeV) ≪ 1 : we'll return to QCD effects in more detail in a bit.]
- So, it is "unlikely" that $\boxed{\text{ejected}}$ quark will hadronize with spectator quarks : instead struck quark will materialize as its $\boxed{\text{own}}$ "jet" of hadrons (again, after emitting soft gluons etc.), roughly moving in direction of momentum transfer from electron
- Whereas, spectator quarks will (separately) form proton $\boxed{\text{"debris"}}$ (of energy/momentum ~ GeV)
- Clearly, proton then "breaks apart" : invariant $\boxed{\text{mass}}$ of (final) hadronic system [i.e, jet formed by ejected (hadronized) quark and remnnant of proton] $\boxed{\gg \text{GeV}}$, hence called (deep) inelastic scattering

— For getting a leading-order formula for this cross-section, it is more convenient to use instead the electron-proton center-of-mass (COM) frame, where both electron and proton are moving rapidly towards each other

— Now, COM energy, $\boxed{\sqrt{s}} = \sqrt{(k+P)^2 = k^2 + P^2 + \underbrace{2k\cdot P}_{\text{in lab frame, this is } 2\underbrace{E_e}_{\text{incoming electron energy}} m_p, \text{ since } P_\mu = (m_p, \vec{0})}}$

$$= \sqrt{m_e^2 + m_p^2 + 2m_p E_e} \approx \boxed{\sqrt{2 m_p E_e}}$$

$\boxed{\gg \text{GeV}}$ if $E_e \gg$ GeV which we will assume henceforth.

— Thus proton has light-like momentum along collision direction, $\boxed{P^2 \simeq 0}$ (again compared to $\boxed{s}$)

— Now, constituents of proton can only acquire a large ($\gg$ GeV) momentum transverse to collision/proton direction by emitting/exchanging hard gluon, but that effect is $\alpha_s\left[(@ \gg \text{GeV}) \ll 1\right]$-suppressed

— So, @ $O\left[\alpha_s^{ⓞ}\right]$, we can set p (initial momentum of struck quark) to be collinear with P (proton momentum, i.e., $\boxed{p \parallel P}$ or

(initial) struck quark $\boxed{p_\mu = \xi \overset{\leftarrow \text{proton}}{P_\mu}}$, where $\boxed{\xi} \in [0, 1]$ is called longitudinal fraction of constituent: clearly $\boxed{p^2 \simeq 0}$ (vs. $\boxed{s}$). Note that we'll return to $\boxed{O(\alpha_s)}$ effects later including those (as just above) giving constituents a transverse "kick" and (hard) gluon exchanges (between ejected quark & debris etc.)", (hard) gluon emission from ejected quark ; finally, virtual gluon exchange between initial & final struck quark (see picture below) [as argued earlier for process(1)]

- Just as a reminder, virtuality of (say) photon exchanged between electron and quark is $Q^2 \gg (\text{GeV})^2$, so time scale of underlying process $\ll 1/\text{GeV}$. Hence, only such hard gluon exchanges—which can operate on similar time scale - can modify the underlying probability of scattering, i.e., soft (~GeV) gluon exchanges are too "slow" in this regard
- Then, we can write

longitudinal fraction of constituent

$$\sigma(e^- p \to e^- \underbrace{X}_{\text{all hadrons}}) = \int d\xi \left[\sigma(e^- q \to e^- q) \text{ for quark with given } \xi\right]$$

can be from photon or Z exchange

(in general)

$$\times \left[\text{probability to find quark with } \boxed{\xi}\right]$$

- However, the above probability is an exclusive (hadronic) quantity, determined by soft ($\lesssim$ GeV) gluon exchanges which bind quarks into proton : again, this is for a specific quark and given $\boxed{\xi}$ [cf. integrals of which might be constrained / fixed, as we will see below].
- Clearly, this probability cannot be computed in QCD perturbation theory : it has to determined (input) from other experiments, i.e., once it is obtained from 1 experiment, it can be used to make other predictions, since it is the same for all processes

- Actually, we need this probability for $\boxed{\text{each}}$ of the constituents of proton with which electron can interact, e.g., antiquark, various flavors of quarks etc., i.e., (in principle) all partons (each species is denoted by $\boxed{f}$, i.e, $f = u, d$, gluon, $\bar{u}$, $\bar{d}$ ...): we denote this probability density by f, i.e

$$\boxed{\text{probability of finding constituent/parton of species } f \text{ with longitudinal fraction between } \xi \text{ and } (\xi + d\xi) \equiv \underset{\swarrow\,(\text{for function})}{f}_{\underset{\curvearrowright\, u,\,d,\,\text{gluon}\ldots}{f}}(\xi)\; d\xi}$$

- $f_f(\xi)$ is called parton distribution function (PDF)

- Thus, we finally have

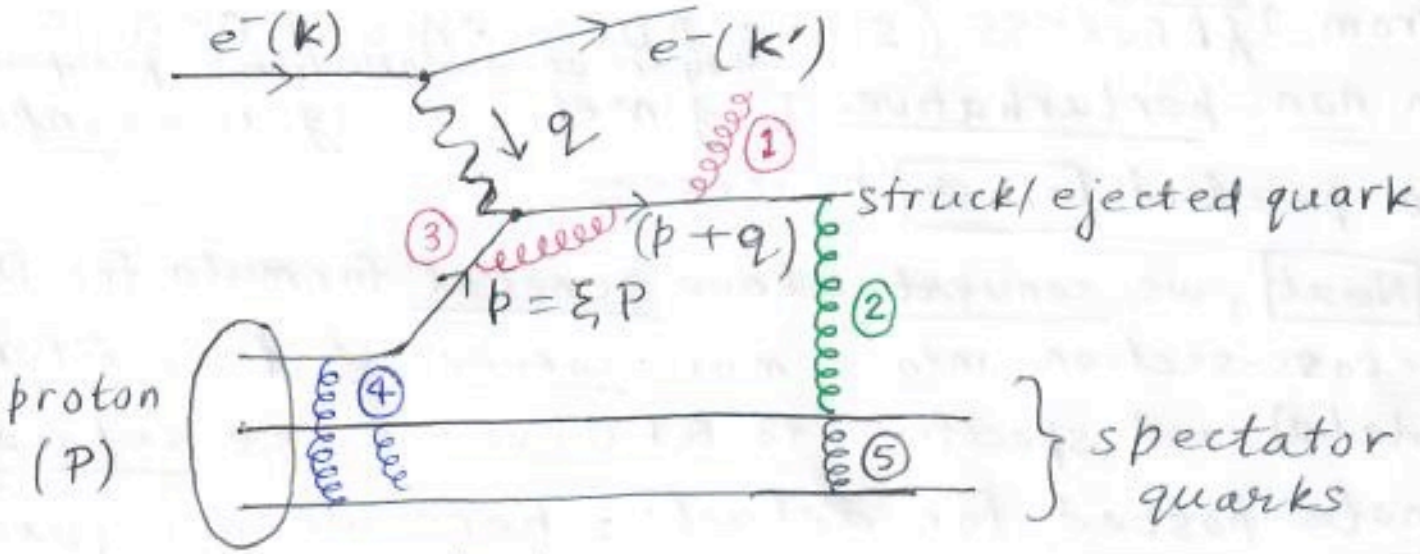
$$\boxed{\sigma[e^-(k)\,\underset{\swarrow\,\text{proton}}{p}(P) \to e^-(k') + X] = \sum_{f=u,d\ldots} \int d\xi \left[f_f(\xi)\right] \sigma\left[e^-(k)\, q_f(\xi P) \to e^-(k')\, q_f(p')\right]}$$

where $(p' + k') = (\xi P + k)$

- Indeed, above formula is equivalent to what was constructed in the $\boxed{\text{parton}}$ model (in 1960's, i.e., before QCD was formulated), assuming $\boxed{\text{free}}$ partons: this was justified with QCD (in 1970's by $\boxed{\alpha_s}$ being small at large momentum scales (i.e., asymptotic freedom), i.e, above is $O[\alpha_s^{\textcircled{0}}]$ formula

- So, $\boxed{\text{onto}}$ $\boxed{O(\alpha_s)}$ corrections from $\boxed{\text{hard}}$ gluon exchange which (as mentioned/indicated above) will modify PDF's (i.e., give partons a large transverse-to-proton direction component of momentum) and electron-quark (underlying) scattering cross-section.

- And, soft gluon exchanges are involved in hadronization: schematically, all these gluon exchanges can be picturized as in



i.e., $\sigma\left[e^-(k) + \overset{\swarrow\text{proton}}{p}(P) \longrightarrow e^-(k') + X\right]$, with QCD correction

$$= \sum_{f=u,d\ldots} \int d\xi \; \underbrace{f_f(\xi)}_{\text{leading-order PDF}} \left[1 + \underset{\nwarrow \text{④ hard}}{O}\left(\frac{\alpha_s}{\pi}\right)\right] \times \text{``}1\text{''}$$

④ soft → leading-order PDF

$\left.\right\}$ hadronization probability for ejected/struck & spectator quarks ①,②,⑤ soft

momentum transfer = k − k′ ↑

$$\underbrace{\sigma\left[e^-(k)\, q_f(\xi P) \rightarrow e^-(k')\, q_f(p' = \xi P + q)\right]}_{\text{no } O(\alpha_s) \text{ here, i.e., (purely) EW}} \left[1 + \underset{\nearrow}{O}\left(\frac{\alpha_s}{\pi}\right)\right]$$

①,③,② hard

- Again, hard gluon exchange/emission is "fast", hence (explicitly) enters cross-section as above; while soft gluon contributions are slow, hence appear more implicitly, i.e., as hadronization probability of 1 & PDF's

— Other processes with proton involving high momentum transfer have similar parton model descriptions, i.e., in QCD at high $Q^2 [\gg (\text{GeV})^2]$, we start with scattering of quarks, gluons... inside proton, whose "initial motion/state" is described by same $f_f(\xi)$ [PDF's] as above in DIS: again, the idea then is to extract PDF's from 1 data on process (since PDF's encode QCD effects in non-perturbative regime they are uncalculable from 1st principles), using it as input to predict other processes.

— Next, we convert above general formula for DIS cross-section into a more convenient (for fitting to data) and specific (to QED) one: see separate note posted for details; here we will just summarize:

$$\boxed{\frac{d\sigma}{dx\,dy}\left(e^-p \to e^-X, \text{via only photon exchange}\right) = \sum_{f = \underbrace{u,\bar{u},d,\bar{d}}_{\text{quarks and antiquark}}} \frac{2\pi\alpha^2 s}{Q^4} \times \left[x\, f_f(x)\, Q_f^2\right]\left[1 + (1-y)^2\right],}$$

where $\boxed{Q^2} = -(k-k')^2 = \textcircled{x}\, y s$; s is the electron-proton COM energy and $\textcircled{y} = 2\, P\cdot(k'-k)/_s$.

— Clearly, k, k' (initial & final electron momentum) and P (initial momentum of proton) are measured directly; so $\boxed{Q^2, x, y \text{ and } s} = (P+k)^2$ entering above cross-section formula are all observable quantities. [Note that $\int d\xi \ldots$ has "disappeared", which is part of usefulness of above form of σ.

Deep inelastic neutrino scattering

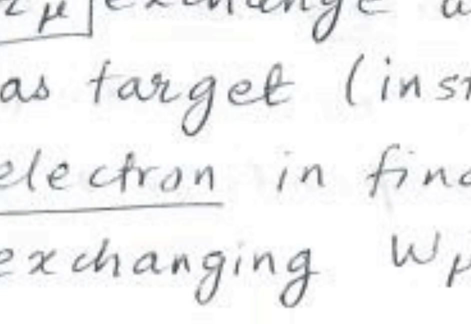
- We saw above that SM prediction for cross-section of electron scattering off of proton depends on a certain combination of PDF's, thus can be used to extract it from data
- Clearly, we need another measurement in order to fix individual PDF's, i.e., cross-section for a different process which depends on a different combination of PDF's
- DI neutrino (ν) scattering provides such a measurement
- DIνS will also nicely illustrate "interplay" of EW and QCD physics
- If we assume, $(\text{GeV})^2 \ll Q^2$ [i.e., $(-)(\text{momentum transfer})^2$] $\ll M_W^2$, then we can (as usual) describe the underlying process, i.e., ν scattering off of q (or $\bar{q}$), via $W^{\pm}_{\mu}$ exchange by Fermi theory (i.e., effective 4-fermion interaction):

$$\mathcal{L}_{\text{effective}} \ni \frac{G_F}{\sqrt{2}} \Big[\underset{\substack{\downarrow \\ \text{creates} \\ \mu^-}}{\bar{\mu}} \gamma_\mu \underbrace{(1-\gamma_5)}_{2P_L} \underset{\substack{\searrow \\ \text{destroys} \\ \nu_\mu}}{\nu_\mu} \Big] \Big[\bar{u} \gamma^\mu (1-\gamma_5) d \Big] + \text{h.c.}$$

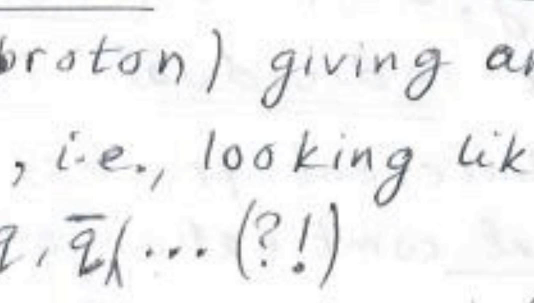
where we have chosen neutrino to be muon type

- Of course, ν_μ can also interact with $q, \bar{q}$ via neutral current ($\boxed{Z_\mu}$, but not photon, exchange), but that possibility can be "vetoed" by insisting on/selecting events where there is an outgoing $\boxed{\mu^-}$: perhaps this is why we didn't use incoming $\nu_{\textcircled{e}}$, since it can interact by $\boxed{Z_\mu}$ exchange with electron inside atom (if that is used as target (instead of proton) giving an energetic electron in final state, i.e., looking like ν_e exchanging W_μ^+ with $q, \bar{q}$(...(?!)

— Interestingly (here's where non-trivial interplay between EW and QCD physics), $\boxed{\nu_\mu}$ can $\boxed{\text{only}}$ scatter off of down quark inside proton (again, restricting to $\boxed{W_\mu}$ exchange), that too LH:



$\boxed{\text{or}}$



i.e.,

ν_μ can interact with anti-up quark, but RH [i.e., anti-particle of u_L : again $\bar{u}$ operator in above $\mathcal{L}_{\text{effective}}$ either creates u_L as in 1st diagram or destroys $(\bar{u})_R$ as in 2nd].

— Similarly, anti-neutrino (muon flavor) can only scatter off of $\boxed{u_L}$ or $\boxed{(\bar{d})_R}$ by $\boxed{W_\mu}$ exchange

— We can obtain cross-section for $\boxed{\text{DI}(\nu)\text{S}}$ by $\boxed{\text{re-casting}}$ (appropriately modifying) that for DI$\boxed{\text{electron}}$S obtained earlier, i.e.,

$$\frac{d\sigma}{dx\,dy}(e^- p \to e^- X) = \sum_f \frac{2\pi \alpha^2 s}{\boxed{Q^4}} \times \left[x f_f(x) Q_f^2\right]\left[1+(1-y)^2\right], \text{ where}$$

$Q^2 = -(\text{momentum transfer})^2 = xys$; $s = (P+k)^2$

$= -(k'-k)^2$

k → incoming electron

k' → outgoing electron

P → (initial) proton momentum

and $y = 2P.(k'-k)/s$

$\boxed{\text{Simple}}$ changes first

$\boxed{(\text{i}).}$ $2\pi\alpha^2 = 2\pi\left(\frac{e^2}{4\pi}\right)^2$ for photon exchange in DIeS

$\longrightarrow 2\pi\left(g^2/4\pi\right)^2$ for W_μ exchange for DIνS

$\boxed{(\text{ii}).}$ $\left[Q_e (\text{charge of electron}) \times Q_f (\text{charge of quark/anti-quark})\right]^{②} \rightarrow \left(1/\sqrt{2}\right)^4$ "charge" i.e. under $W^\pm_\mu$ from $(\sigma_\pm/2)$

$\boxed{(\text{iii}).}$ $\boxed{1/Q^4}$ from [photon propagator (with momentum = that transferred, i.e. $k'-k$)$]^2 \longrightarrow \boxed{1/M_W^4}$ from W_μ propagator (dropping momentum in it vs. M_W)

$\boxed{(\text{iv}).}$ $\boxed{\text{No}}$ $(1/2)$ from averaging over polarizations of incoming ν (or $\bar{\nu}$) since its always LH (or RH), cf. electron is 50% LH(RH). Averaging over incoming quark/anti-quark applies to $\boxed{\text{both}}$ cases.

$\boxed{(\text{v}).}$ Related to (iv). above: for electron, we get factor of 2 in cross-section due to LH & RH contributing equally, but for ν (or $\bar{\nu}$), only LH (or RH) is present [in detail, $(e^-)_L (u)_L$ is same as $(e^-)_R u_R$ scattering etc.

So, net effect of above 5 changes is that

$$\boxed{Q_f^2\, 2\pi\alpha^2/Q^4} \longrightarrow \boxed{G_F^2/\pi} \quad \text{again } \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

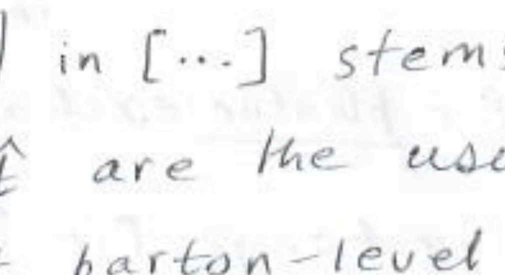
$\boxed{\text{v.}}$ More subtle change for rest of the factor [other than "$s\,x$" which is from kinematics, hence stays same], i.e., $\boxed{f_f(x)\left[1+(1-y)^2\right]}$.

- In order to understand how to apply this to DIνS, we have to remind ourselves (see other posted note on DIeS) that

① in $[\cdots]$ above "comes from" $\hat{s}^2$, while $\boxed{(1-y)^2}$ in $[\cdots]$ stems from $\boxed{\hat{u}^2}$, where $\hat{s}, \hat{u}, \hat{t}$ are the usual $\boxed{\text{Mandelstam}}$ variables, but at parton-level (cf. $\boxed{S}$ is at proton-level):

$\boxed{\hat{s}} = (k+p)^2 \simeq 2\,p\cdot k$ (neglecting masses)
↳ incoming quark/anti-quark momentum
$= 2\,P\xi\cdot k = \xi S$, i.e.,

$\boxed{\text{no}}$ dependence on θ_{CM}, which is scattering angle in the electron-parton COM frame:



(with $p+k = p'+k'$)

$$\boxed{\hat{t}} = (k'-k)^2 = q^2 = (-)Q^2 = -\hat{s}\,(1-\cos\theta_{CM})/2$$

(which clearly shows that $\hat{t}<0$ and $|\hat{t}| \leq s$, latter relation was used in earlier note)

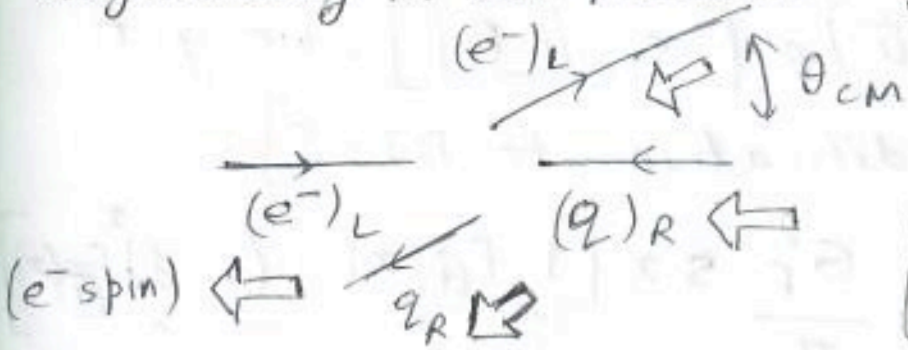
$$\boxed{\hat{u}} = (p' - k)^2 = -\hat{s}\left(1 + \cos\theta_{CM}\right)/2 = -\hat{s}\left(1 + \hat{t}/\hat{s}\right) = -\hat{s}\left[1 - Q^2/s\xi_1\right]$$

– Now $\hat{s}^2$, $\hat{u}^2$ in the form of "1" and "$(1-y)^2$" come from Dirac matrix algebra [i.e., are specific to <u>partons</u> being <u>fermions</u>]: they "encode" helicity information as follows (of course, helicity ≃ chirality in relativistic limit here)

– Take the case of "<u>LR</u>" scattering, i.e.,

$$(e^-)_{\textcircled{L}} + [q\,(\text{or } \bar{q})]_{\boxed{R}} \longrightarrow (e^-)_L + [q\,(\text{or } \bar{q})]_R$$

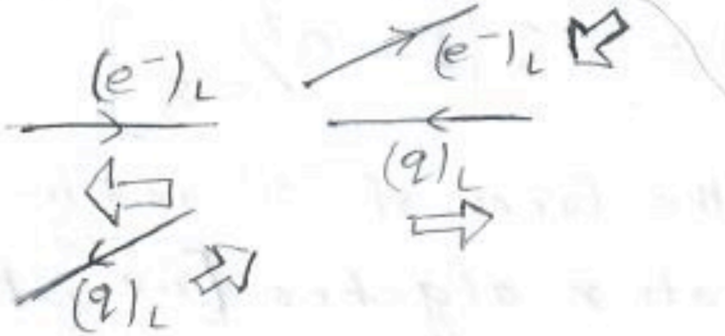
(again, <u>helicities</u> do <u>not</u> <u>change</u> in this process, i.e., "γ_μ" structure of interaction), where <u>orientation of</u> $\boxed{\text{spins}}$ (a ka what was done for forward-backward asymmetry in EW phenomenlogy discussion/note!):



[total spin component along direction of motion of initial/final state $= \underbrace{(-1/2)}_{(e^-)_L} + \underbrace{(-1/2)}_{(q)_R} = -1$]

shows that <u>amplitude</u> should <u>vanish</u> in <u>backward</u> direction [i.e., $\theta_{CM} = \pi$], suggesting that $\boxed{\text{"LR"}}$ contribution [similarly $\boxed{\text{"RL"}}$, i.e, $(e^-)_{\boxed{R}} + q_{\textcircled{L}} \rightarrow \ldots$] is $\textcircled{\hat{u}^2} \propto \boxed{(1+\cos\theta_{CM})^2}$, thus $\boxed{(1-y)^2}$ in DIeS cross-section

– Whereas, Ⓛⓛ [i.e., $(e^-)_L (q)_L \rightarrow (e^-)_L (q)_L$] has $\boxed{\text{no}}$ such "preference" for $\boxed{\theta_{CM}}$ <u>dependence</u> based on orientation of spins (since total spin-component = 0 along direction of motion)



$$\underbrace{\left(-\tfrac{1}{2}\right)}_{(e^-)_L} + \underbrace{\left(+\tfrac{1}{2}\right)}_{(q)_L}$$

Thus, while not a "proof" (for that one has to work out Dirac matrix algebra, i.e., insert P_{LR} (projection) operators in calculation of Peskin, Schroeder, section 5.4), we understand that $\textcircled{LL}$ [similarly $\textcircled{RR}$, i.e., $(e^-)_R (q)_R \to (e^-)_R (q)_R$] contributions correspond to $\boxed{\hat{s}^2}$ (again, no θ_{CM} dependence) or $\boxed{\text{"1"}}$ term in DIeS formula

- Since incoming $(\nu)_{\textcircled{L}}$ interacts only with d_L [i.e., $\textcircled{LL}$] and $(\bar{u})_R$ [i.e., $\boxed{LR}$], we get [combining (i)-(vi) modifications to DIeS]:

$$\frac{d^2\sigma}{dx\,dy}\left(\nu_\mu p \to \mu^- X\right) = \underbrace{\frac{G_F^2}{\pi}}_{\substack{\text{replaces} \\ Q_f^2\, 2\pi\alpha^2/Q^4 \\ \text{as per (i)-(iv) above}}} s\,x\Big[\underset{\substack{\uparrow \\ \text{from LL} \\ \propto \hat{s}^2}}{1 . f_{\textcircled{d}}(x)} + \underset{\substack{\uparrow \\ \text{from LR} \\ \propto \hat{u}^2}}{(1-y)^2 f_{\boxed{\bar{u}}}(x)}\Big]$$

- Similarly, incoming $(\bar{\nu})_{\boxed{R}}$ interacts with u_L [i.e., RL] or $(\bar{d})_R$ [i.e., RR] giving

$$\frac{d^2\sigma}{dx\,dy}\left(\bar\nu_\mu\, p \to \mu^+ X\right) = \frac{G_F^2}{\pi} s x \left[f_{\boxed{\bar d}}(x) + (1-y)^2 f_{\textcircled{u}}(x) \right]$$

$\uparrow$ from RR $\propto \hat{s}^2$ $\quad$ $\uparrow$ from RL $\propto \hat{u}^2$ of earlier

— **Comparing** cross-section formulae for **DIeS** and (DIνS) just above, we see that (as promised) the 2 processes depend on different combinations of PDF's

— Note that to the extent that proton is made up of "mostly" quarks, i.e., not anti-quarks, we expect that (differential) cross-section for $\boxed{\nu_\mu}$ scattering off of proton should be roughly constant in $\boxed{y}$ (since y-dependence comes from $\bar u$ PDF), while falling-off as $\boxed{(1-y)^2}$ for $\boxed{\bar\nu_\mu}$ (since we can neglect constant piece from $\bar d$ PDF)

Properties of PDF's of quarks & anti-quarks

— As mentioned above, PDF's can be extracted from data on DIeS and DIνS

— How about **gluon** PDF (which we can't get directly from DIS, since leptons don't interact directly with gluons inside proton)?

— The point is that (as indicated earlier), at $O[\alpha_s^{\textcircled{0}}]$, PDF's of quarks/anti-quarks entering DIS crosssections depend only on x, i.e., are independent of $\boxed{Q^2}$ (this is called "Bjorken scaling")

- However, at higher-order in (perturbative) QCD, the involved PDF's change (slowly) with $\boxed{Q^2}$: this is the hard gluon emissions/exchange part of corrections labelled ④ in earlier picture : clearly then PDF's of gluons and quarks/anti-quarks are "coupled" at $O(\alpha_s)$
- Thus, measurement of above "violation" of Bjorken scaling enables determination of gluon PDF's (which are needed, for example, for calculation of gluon fusion production cross-section of Higgs boson).
- Also, u and d (vs. $\bar{u}, \bar{d}$ or gluon) carry most of proton's momentum at least at large $\boxed{x}$, based on proton being a "uud" bound state
- Indeed, PDF's must be normalized so as to reflect above quantum numbers of proton; again proton = uud ("valence" partons) + gluon & $q\bar{q}$ (in principle, all flavors) pairs ("sea" partons)
- Thus, proton should have an "excess" of 2 up & 1 down quarks over $\bar{q}$, i.e.,

$$\int_0^1 dx \left[f_u(x) - f_{\bar{u}}(x) \right] = 2 \; ; \; \int_0^1 dx \left[f_d(x) - f_{\bar{d}}(x) \right] = 1$$

(again, PDF's are after all probability densities for finding various partons inside proton

- So far, we have (implicitly sometimes) discussed

PDF's of proton: it is straightforward to extend these arguments to $\boxed{\text{neutron}}$, i.e., $\textcircled{udd}$ bound state, with "extra" gluon & $q\bar{q}$ pairs.

- Clearly (again $p \to n$ by simply $d \Leftrightarrow u$), we get (approximately) $f_u^{\textcircled{n} \leftarrow \text{neutron}}(x) = f_d(x)$, i.e., for proton; $f_d^{\textcircled{n}}(x) = f_u(x)$ ... similarly, for anti-proton, i.e., $f_u^{\bar{p} \leftarrow \text{anti-proton}}(x) = f_{\bar{u}}(x)$, i.e., for proton

- Finally, since $\boxed{\text{total}}$ momentum of proton must come from those of its various constituents/partons, we have a "completeness relation":

$$\int_0^1 dx \underbrace{\textcircled{x}}_{\substack{\text{fraction of momentum}\\ \text{carried by parton}}} \left[f_u(x) + f_d(x) + f_{\bar{u}}(x) + f_{\bar{d}}(x) + \underbrace{f_g(x)}_{\text{PDF of gluon}} \right] = 1$$

(neglecting tiny strange, charm quark etc. content of proton)