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Detailed Electroweak (EW) theory/ unification: part I ($\boxed{\text{bosonic}}$ sector)

— Next, we supply detailed formula corresponding to overview given earlier in the following order/outline:

- unbroken gauge symmetry/Lagrangian
- Higgs mechanism to give gauge boson masses; identify "bottomline" [i.e., mass] eigenstates [in particular $\boxed{\text{photon}}$ and Z as admixtures of original neutral gauge bosons] and physical/leftover scalar (Higgs) boson
- add 1 generation of fermions; determine how they couple to above gauge boson mass eigenstates (in particular, to "extra" Z boson)
- $\boxed{\text{Yukawa}}$ couplings of fermions to Higgs field, giving fermion masses

"Starting" $\boxed{\text{EW}}$ gauge sector

— We already did an overview in previous note, so here we just fill in details

— $\boxed{\text{Gauge group}}$ is $\boxed{SU(2)_L \times U(1)_Y}$

hypercharge

acts only on left chirality of SM fermions

— $\boxed{\text{generators}}$ of $SU(2)_L$: we will only need the doublet/fundamental representation for SM fermions/Higgs field, i.e., $\frac{\sigma^a}{2}$ $(a=1,2,3)$

— $\boxed{\text{gauge bosons}}$ of $SU(2)_L$ are $\boxed{W_\mu^a}$, while hypercharge gauge boson will be denoted by B_μ

— It is convenient to form combinations of $\sigma_{1,2}$:

$\boxed{\sigma_{\pm}}$ (not Hermitian) $= \frac{1}{\sqrt{2}}(\sigma_1 \pm i\sigma_2)$,

corresponding to gauge bosons $\boxed{W_\mu^{\pm}} = \frac{(W_1 \pm i W_2)}{\sqrt{2}}$

(Note that structure constants in this basis are not completely antisymmetric.)

[Also, recall that $U(1)_Y$ $\boxed{\text{commutes}}$ with $SU(2)_L$ so that its generator is simply $\boxed{Y\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ in the doublet representation for SM fermion & Higgs field, with hypercharge Y for entire doublet, i.e., both components.]

— $\boxed{\text{Pure}}$ gauge Lagrangian $= -\frac{1}{4} B^{\mu\nu} B_{\mu\nu}$
$- \frac{1}{4} W_a^{\dagger\mu\nu} W_{\mu\nu}^a$,

where $\boxed{B_{\mu\nu}} = \partial_\mu B_\nu - \partial_\nu B_\mu$

and $\boxed{W_{\mu\nu}^a} = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g f_{abc} W_\mu^b W_\nu^c$ (non-abelian)

$a = \underbrace{+,-}_{\text{new basis for } \sigma_{1,2}}, 3$

[Note that since f_{abc} are not real in new basis for σ^a's, we need to use "dagger" of W_a^μ.]

$\boxed{\text{Higgs}}$ mechanism (spontaneous EW gauge symmetry breaking): basically this is a suitable $\boxed{\text{generalization}}$ of U(1) only (as in lecture and HW 4) and SU(2) only (HW 7.1.2) to combination of $\boxed{SU(2)}$ and $\boxed{U(1)}$

- We introduce a $\boxed{\text{complex}}$ scalar doublet of $SU(2)_L$ with Y (hypercharge) $= +\frac{1}{2}$, denoted by

$$\boxed{\Phi} = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} : (\underbrace{2}_{SU(2)_L}, \underbrace{\tfrac{1}{2}}_{U(1)_Y})$$ (again, both ϕ_+ and ϕ_0 are complex),

where sub-scripts "+" and "0" on ϕ's denote their electric charges (see later)

- $\boxed{\text{Lagrangian}}$ for ϕ is simple generalization of U(1) [single complex scalar] case:

$$\mathcal{L}_\Phi = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2,$$ where

$$\Phi^\dagger \Phi = (\underbrace{\phi_+^\dagger \ \phi_0^\dagger}_{\text{"dagger"}}) \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} = \underbrace{\phi_-}_{= \phi_+^\dagger} \phi_+ + \phi_0^\dagger \phi_0$$

and $$D_\mu \Phi = \Big[\partial_\mu + i g \underbrace{\frac{\sigma_a}{2}}_{\text{generator of (matter) representation (doublet here)}} W_\mu^a + i g' \underbrace{\left(\frac{1}{2}\right)}_{\text{in general Y of matter field}} B_\mu\Big] \Phi$$

$= (2 \times 1$ column vector$) \begin{pmatrix} \partial_\mu \phi_+ \\ \partial_\mu \phi_0 \end{pmatrix} +$ (continued on next page)

$$\frac{i}{2}\begin{bmatrix} g W_\mu^3 + g' B_\mu & \sqrt{2}\, g W_\mu^+ \\ \sqrt{2}\, g W_\mu^- & -g W_\mu^3 + g' B_\mu \end{bmatrix}\begin{pmatrix}\phi_+ \\ \phi_0\end{pmatrix}$$

– As usual, we assume $\boxed{\mu^2 < 0}$ so that $\Phi_{\text{classical}} = \langle 0 | \Phi | 0 \rangle$ $\boxed{\text{VEV}}$ $= \frac{v}{\sqrt{2}}\, \chi$ ← constant doublet with $\chi^\dagger \chi = 1$
and $\boxed{v = \sqrt{-\mu^2/\lambda}}$

– Using the $SU(2)_L \times U(1)_Y$ symmetry, a general form of χ can be rotated into $\boxed{\begin{pmatrix}0\\1\end{pmatrix}}$, so we assume (without loss of generality) $\langle 0 | \Phi | 0 \rangle = \begin{pmatrix}0 \\ v/\sqrt{2}\end{pmatrix}$: this is of course consistent with above assignment of electric charge 0 to lower component of Φ [see later for more on this choice].

– Thus, symmetries generated by $\boxed{T_{1,2}}$ (generators for general representation) are broken, since $\sigma_{1,2}$ (generators for Φ) $\Phi_{\text{classical}} \neq 0$ (again, see HW 5.3.2). Hence, we expect corresponding gauge bosons ($W_\mu^\pm$) to be massive (as we will explicitly show below)

– Similarly, symmetries generated by $\boxed{T_3}$ [3rd component of $SU(2)_L$] and $\boxed{Y}$ are separately broken, but their combination,

$\boxed{Q}$ (which will be identified with electric charge) $\boxed{= T_3 + Y}$ is un broken (i.e., photon — combination of W_3^μ & B^μ — is massless):

$\sigma_3\, \Phi_{classical} \neq 0$; $Y\, \Phi_{classical} \neq 0$; $\left[Q\left(=\frac{\sigma_3}{2}+Y\right)\right]\Phi_{classical} = 0$

$$Y \to \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad Q \to = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

← note this is Q of ϕ_0

Note: $\sigma_- \Phi_{classical} = 0$, ($\sigma_-$: only non-zero in 1st column)
but σ_- is not Hermitian, so $\exp(-i\beta_- \sigma_-)$ is not a unitary transformation

— So, as expected/promised, we have

$$\boxed{SU(2)_L \times U(1)_Y \longrightarrow U(1)_{EM}}$$

with massive $\boxed{W^\pm_\mu}$ and $\boxed{Z^\mu}$ (combination of B_μ and W^3_μ which is orthogonal to photon) vs. massless photon (denoted by A_μ)

— So we now "understand" electric charge (Q) assignments (sub-scripts) given earlier: ϕ_+ for upper component of Φ due to $Y = +1/2$ & $T_3 = +1/2$ vs. ϕ_0 for lower component due to $Y = +1/2$, but $T_3 = -1/2$. Similarly, $\boxed{W_\mu^\pm}$ due to $Y = 0$ [$U(1)_Y$ commutes with $SU(2)_L$ so that W_μ^a are not "charged" under $U(1)_Y$] and $T_3 = \pm 1$, since W_μ^a's are triplet/adjoint under $SU(2)_L$ [see discussion of pion-nucleon system, where $\pi^{\pm}$ were shown to have T_3 eigenvalues ± 1]

— Just to complete the story, if we choose instead $\Phi_{classical} = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}$, then we "re-label" Φ as $\begin{pmatrix} \phi_0 \\ \phi_- \end{pmatrix}$, still with $\boxed{Y = +1/2}$, but now $Q = T_3 - Y$. Alternately, we stick to $\boxed{Q = T_3 + Y}$, but choose $\boxed{Y = -1/2}$ for $\Phi = \begin{pmatrix} \phi_0 \\ \phi_- \end{pmatrix}$

— Next, we work out $\boxed{\text{gauge}}$ boson masses in order to verify above expectations

- Expressing (in linear representation) $\Phi = \begin{pmatrix} \phi_+ \\ \frac{v + H + i\zeta}{\sqrt{2}} \end{pmatrix}$ (with zero VEV's for H, ζ and ϕ_+), we get the following mass^2 matrix for $W^{\mu}_{a=1,2,3}$ and B_μ (as usual putting Φ to its VEV and picking gauge field from both "D_μ"'s in $\mathcal{L}_\Phi$)

$$\mathcal{L}_{gauge\ mass^2} = \left\{ \frac{i}{2} \begin{bmatrix} g W^3_\mu + g' B_\mu & \sqrt{2}\, g W^+_\mu \\ \sqrt{2}\, g W^-_\mu & -g W^3_\mu + g' B_\mu \end{bmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \right\}^{\dagger} \ \text{row}$$

$$\times \underbrace{\left\{ \cdots \right\}}_{\boxed{\text{same}}} \leftarrow \text{column}$$

$$= \frac{1}{4} g^2 v^2 W^+_\mu W^\mu_- + \frac{1}{8} v^2 \left(-g W^3_\mu + g' B_\mu \right)^{\text{"}2\text{"}}$$

— Clearly, $\boxed{M_W = \frac{1}{2} g v}$ (since $\mathcal{L}_{W^\pm mass^2} = M_W^2 W^+_\mu W^\mu_-$, like for $\boxed{\text{complex}}$ scalar)

- Whereas, in $\boxed{\text{neutral}}$ sector, we define [note this is a orthogonal transformation of W_3^μ, B^μ so that kinetic terms for A_μ, Z_μ (new basis) are still canonical]

$$\boxed{Z_\mu = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu}$$
$$\boxed{A_\mu = \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu},$$

where $\boxed{\tan\theta_W = g'/g}$ so that $\boxed{M_A = 0}$ (photon)

and $\boxed{\begin{aligned} M_Z &= (1/2)\, v\sqrt{g^2+g'^2} \\ &= M_W/\cos\theta_W \end{aligned}}$ (since $\mathcal{L}_{Z\,\text{mass}^2} = \left(\frac{1}{2}\right) M_Z^2 Z_\mu Z^\mu$, like for real scalar)

(again, as expected from one combination of T_3 & Y generators being unbroken)

- As a $\boxed{\text{sanity}}$ check, let's make sure $\boxed{A_\mu}$ does couple to charge $\boxed{Q = T_3 + Y}$ (in this process we will also determine "charge" of a field under (Z_μ)):

D_μ [neutral or abelian part of $SU(2)_L \times U(1)_Y$]

$= \partial_\mu + ig\, T_3 W_\mu^3 + ig' Y B_\mu$ [where T_3 and Y are charges of fields on which D_μ acts]

$= \partial_\mu + ig\, T_3 (\cos\theta_W Z_\mu + \sin\theta_W A_\mu)$ ← using inverse of above
$\quad + ig' Y(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu)$ ← transformation

$$= \partial_\mu + i\, e\, A_\mu \underbrace{(T_3 + Y)}_{\text{electric charge, } Q} + i\, g_Z Z_\mu \underbrace{(T_3 - Q\sin^2\theta_W)}_{\text{"}Q_Z\text{"}},$$

using $\tan\theta_W = g'/g$ and $Q = T_3 + Y$: we see that QED coupling constant, $\boxed{e = gg'/\sqrt{g^2+g'^2}}$, while that of Z_μ is $\boxed{g_Z = \sqrt{g^2+g'^2}}$, with charge under Z, $\boxed{Q_Z = T_3 - Q\sin^2\theta_W}$

— Thus, just to belabor the point, massless combination of T_3 & B_μ does indeed couple to electric charge

[Also, we will use above coupling to Z for discussing phenomenlogy of EW sector.]

— Just for the sake of completeness, let's briefly discuss $\boxed{\text{scalar}}$ (spin-0) modes of EW sector

— Let's first do counting of scalar physical degrees of freedom (d.o.f.), i.e., in unitary gauge: we start with $\boxed{4}$ real scalar fields (in 1 complex doublet Φ), but 3 of these are eaten by (massive) $W^\pm$ & Z (becoming their longitudinal) polarizations (as usual) $\Rightarrow$

we have 1 physical scalar field leftover (called Higgs boson): at linear order, it is $\boxed{H}$ of linear representation for Φ shown above

— We can explicitly and simply show above eating (going to unitary gauge) using Φ's radial representation:

we did this for $U(1)$ in lecture; maybe you did it for $SU(2)$ in HW 5.4 and 7.1.2; now it's $SU(2)\times U(1)$ as in HW 8.2

- However, just like for $U(1)$ case discussed in lecture, for (manifest) renormalizability, it is better to use $\boxed{R_\xi}$ gauge, which includes the unphysical/would-be NGB's (again, eaten by massive gauge bosons), along with scalar/time-like polarizations of massive gauge bosons (also unphysical).
- $\boxed{\text{To get to } R_\xi \text{ gauge}}$ [again, just like for $U(1)$], we start with linear representation for Φ; identify would-be NGB's by looking for quadratic/$\boxed{\text{mixing}}$ term with 1 gauge boson and 1 scalar mode;

$$i M_W \left(W_\mu^+ \partial^\mu \phi_- - W_\mu^- \partial^\mu \phi_+ \right) - M_Z Z_\mu \partial^\mu \underset{\text{part of } \phi_0}{\xi} \left(= \mathcal{L}_{mix}\right),$$

which shows (as expected) that $\boxed{Z_\mu}$ eats $\boxed{\xi}$ while $W_\mu^\pm$ eats $\phi_\pm$ [just generalization of $U(1)$ to $SU(2)\times U(1)$]
- As with $U(1)$ case, we choose gauge-fixing terms to "get rid of" above mixing:

$$\mathcal{L}_{gauge\text{-}fixing} = -\frac{1}{\xi} \left| \partial_\mu W^{\mu +} + i \xi M_W \phi_+ \right|^2 - \frac{1}{2\xi} \left(\partial_\mu Z^\mu + \xi M_Z \xi \right)^2$$

(which combine with above mixing terms to give total divergence, thus irrelevant).

- Finally, we have the would-be/unphysical NGB propagators:

$$i\,\Delta^{(\phi_+)}(k) = \frac{i}{k^2 - \xi M_W^2} \quad \& \quad i\,\Delta^{(\varsigma)}(k) = \frac{i}{k^2 - \xi M_Z^2}$$

["corresponding to" gauge boson propagators:

$$i\,D^{(W)}_{\mu\nu}(k) = \frac{-i}{k^2 - M_W^2 + i\epsilon}\left[g_{\mu\nu} - \frac{(1-\xi)\,k_\mu k_\nu}{k^2 - \xi M_W^2}\right]$$

$+\ W \to Z$,

which, as usual, scale as $\sim 1/k^2$ for $k \to \infty$, for fixed ξ, thus clarifying renormalizability

$\boxed{\text{and}}$ Higgs boson (physical) propagator

$$i\Delta^{(H)}(k) = \frac{i}{k^2 - M_H^2}, \text{ with } \boxed{M_H^2 = 2\lambda v^2}$$

- As an aside, we comment on relation between W & Z masses based on $\boxed{\text{custodial}}$ symmetry

- It is convenient to define ρ parameter as

$$\boxed{\rho \equiv M_W^2 / (M_Z^2 \cos^2\theta_W)}, \text{ where } \cos^2\theta_W = \frac{g^2}{g^2 + g'^2}$$

$\boxed{= 1}$ (at tree-level) is SM theory prediction (i.e., with W, Z masses originating from VEV of complex scalar doublet)

[Indeed, experimentally $\rho = 1$ to a good approximation]

- Natural question is whether $\rho = 1$ even if

EWSB is done by scalar field in a $\boxed{\text{different}}$ representation of SU(2) ?!

- Answer is (in general) "No", i.e., $\boxed{\rho = 1}$ is rather specific to the case of doublet as follows

- For a complex doublet (Φ), it is clear [see HW 8.1 for potential subtlety] that (at renormalizable level), $\boxed{V(\Phi)}$ is solely a function of $\Phi^\dagger \Phi$ (due to gauge invariance) $= \sum_{i=1}^{4} |\phi_i|^2$, where ϕ_i are (4) real scalar fields (forming complex doublet)

- Thus (see HW 5.4 & 7.1.2), $V(\Phi)$ has larger than $\boxed{SU(2)}$ global symmetry, i.e., SO(4) [$\sim SU(2) \times SU(2)$] rotating 4 (real) ϕ_i's into each other: this is broken down to $\boxed{SO(3)}$ ($\sim SU(2)$) by VEV of one of the real ϕ_i's, i.e., rotations among other 3 ϕ_i's is still a symmetry [see HW 5.3/7.1.1 for $SO(3) \to SO(2)$ version]

- The $\boxed{\text{broken}}$ combination of (2) SU(2)'s of this larger SO(4) symmetry is gauged, thus corresponding gauge bosons ($W^\mu_{1,2,3}$) get masses

- However, the unbroken $\boxed{SO(3)}$ or SU(2), under which $\boxed{W_\mu\text{'s}}$ transform as a triplet, ensure that they get identical mass terms; that's why the unbroken SU(2) is called "custodial" symmetry/isospin

- Of course, the full $\overset{EW}{}$ gauge symmetry is $SU(2)_L \times U(1)_Y$, which is broken down $\overset{\text{by } \Phi \text{ VEV}}{}$ to $U(1)_{EM}$, i.e., $\boxed{W^{\mu}_{3}}$ mixes with $\boxed{B_\mu}$ (hypercharge gauge boson), since Φ $\overset{VEV}{}$ has non-zero hypercharge [while $W_{1,2}$ have no such counterpart]

- Hence, $M_{W3} \xrightarrow{g' \neq 0} M_Z \neq M_{W_{1,2}}$ (or M_W) which is why ρ is defined with factor of $\cos^2\theta_W$ (i.e, that "takes care of" above $W^{\mu}_3 - B^{\mu}$ mixing): gauging $\overset{\text{hypercharge thus}}{}$ breaks custodial symmetry $\overset{\text{even in SM}}{}$ [again $\boxed{M_Z \to M_W}$ as $\boxed{g' \to 0}$ ($\cos\theta_W \to 1$), i.e., we remove hypercharge gauging, thus there is no W^{μ}_3 mixing with B_μ]

- In HW 8.4, you are asked to compute $\boxed{\text{shift}}$ in ρ parameter due to contribution of a $SU(2)$ triplet VEV (which doesn't enjoy above custodial protection, thus is expected to give $\rho \neq 1$)

[Similarly, integrating out (heavy) new particles can give non-renormalizable terms, e.g., "Φ^{6}" in $V(\Phi)$, which are $\boxed{\text{not}}$ functions of simply $\boxed{\Phi^\dagger \Phi}$, thus also breaking custodial symmetry]