Phenomenlogy of $\boxed{\text{EW}}$ theory/sector of SM : part $\boxed{\text{II}}$ (flavor or generational mixing; theory/overview) ①

[References for details : section 11.3 of Cheng, Li; section 20.3, page 713 onwards, of Peskin, Schroeder; section 15.34 of Lahiri, Pal (for "theory" of flavor); section 12.2 of Cheng, Li for observables] (out of 3)

– Most of $\boxed{\text{previous}}$ discussion assumed only $\boxed{1}$ generation of SM leptons & quarks or neglected mixing among generations : let's take that into account now

– Obviously, we have

$$\underbrace{\begin{pmatrix}\nu_L\\ e^-_L\end{pmatrix};\ e^-_R \text{ (of before)}}_{1^{st}\text{ generation}} \rightarrow \begin{pmatrix}\nu_{e_L}\\ e^-_L\end{pmatrix};\ e^-_R + \underbrace{\overbrace{\begin{pmatrix}\nu_{\mu_L}\\ \mu^-_L\end{pmatrix}}^{\text{muon}};\ \mu^-_R + \overbrace{\begin{pmatrix}\nu_{\tau_L}\\ \tau^-_L\end{pmatrix}}^{\text{tau}};\ \tau^-_R}_{2^{nd}\ \&\ \boxed{3^{rd}}\text{ generations}}$$

Similary (not showing L, R structure for simplicity),

$$\underbrace{u,\ d}_{1^{st}\text{ generation}} \longrightarrow u,\ d + \underbrace{\underset{(c,\ s)}{\text{charm, strange quarks}} + \underset{(t,\ b)}{\text{top, bottom quarks}}}_{2^{nd}\ \&\ \boxed{3^{rd}}\text{ generations}}$$

– $\boxed{\text{Gauge}}$ quantum numbers of 2nd & $\boxed{3^{rd}}$ generations are identical to those of 1st generation (which were discussed in detail earlier), but masses get higher. (2)

— Even though it looks like we are simply making "copies" here, actually, there's very interesting, new phenomenlogy we get in $\boxed{\text{quark}}$ sector (i.e., beyond the processes discussed before), i.e., "mixing" between generations and $\boxed{\text{CP}}$ violation (i.e., decay widths of particle and its anti-particle can be different). However, in the SM $\boxed{\text{lepton}}$ sector, we do not have analogous effects.

— In order to understand these phenomena (including their absence in lepton sector), we need to develop first the "theory" of flavor/generations derived from structure of SM fermion Yukawa couplings / masses (upon Higgs field getting VEV)

$\boxed{\text{Theory of flavor}}$

— Let's begin with $\boxed{\text{SM}}$ lepton sector: we have

$$\mathcal{L}_{\text{Yukawa (lepton)}} = -(h_e)_{\boxed{ij}} \bar{L}^{\boxed{\text{weak}}}_{i} \Phi e^{\boxed{\text{weak}}}_{R j} + \text{h.c.}$$

$\boxed{i=1,2,3}$ (generation index) ←

where we have "suppressed" $SU(2)_L$ indices [contraction to obtain $SU(2)_L$ singlet etc.], since that's not so relevant here. So, $\boxed{i, j = 1, 2, 3}$

denote generation index instead.

— And, the superscript "weak" on L and e_R implies that these are gauge (specifically weak interaction) eigenstates / basis, i.e., the $\boxed{\text{EW}}$ gauge couplings are diagonal in generation space in this basis (i.e., no "mixing" of generations (i,j) via gauge interactions for these states, so just "tack on" index i on each of 2 fermions involved in Feynman rules of before).

— Recall that $L = \begin{pmatrix} \nu_{e_L} \\ e^-_L \end{pmatrix}$, while e_R is $SU(2)_L$ singlet

— Now, (h_e) is a general (complex), off-diagonal $\boxed{3 \times 3}$ matrix (again, in generation space), which can be diagonalized by a bi-unitary transformation, i.e., multiplying from left and right by different unitary matrices (see Cheng, Li pages 357-359)

$$\boxed{h_e^{\boxed{\text{diag.}}} = E_L^\dagger \, h_e \, E_R} \quad (E_{L,R} \text{ are unitary matrices})$$

or $h_e = E_L \, h_e^{\boxed{\text{diag}}} \, E_R^\dagger$

— So, we do a $\boxed{\text{change}}$ basis (why new one is called "mass" will be clear in a bit, if not already so!):

$$\boxed{e_R^{\text{(mass)}} = \underbrace{E_R^\dagger}_{3\times 3 \text{ matrix}} \, \underbrace{e_R^{\boxed{\text{weak}}}}_{3\times 1 \text{ column}}} \quad \& \quad \boxed{L^{\text{(mass)}} = E_L^\dagger \, L^{\boxed{\text{weak}}}}$$

— $\boxed{\text{Note}}$, we choose to rotate (again, in generation space) $\boxed{\nu_{e_L}}$ & e^-_L identically, i.e., E_L rotates entire doublet [just to be clear, this rotation is independent of/unrelated to $SU(2)_L$ gauged rotations we studied earlier which can "change" ν_e to e^-]

— Clearly, charged (weak) current interactions, i.e., $\boxed{W^\pm}$ couplings, remain diagonal even in new/mass basis:

$$\underbrace{\overline{(e^-_L)}^{\boxed{\text{weak}}}_i}_{\text{creates } e^-} \; \underbrace{\delta_{ij}}_{\mathbb{1}} \; \underbrace{\cancel{W}^-}_{\text{destroys } W^-} \; \underbrace{(\nu_L)_j}_{\text{destroys } \nu} = \overline{(e^-_L)}^{\text{mass}} \underbrace{E_L^{\dagger} \, \mathbb{1} \, E_L}_{\mathbb{1}} \nu_L^{\text{mass}}$$

crucial

[again, $W^\pm$ couplings are "off-diagonal", i.e., connect ν_e to e, in $\boxed{SU(2)}$ space in any basis, but that's not focus here.]

(Contrast, above 2-related-points with what we will have for quark sector below.)

— Neutral current (photon, Z exchange) are also "generation index preserving", e.g.,

$$\overline{(e^-_L)}^{\boxed{\text{weak}}}_i \; \delta_{ij} \; \underbrace{\cancel{Z} \left(-\tfrac{1}{2} + \sin^2\theta_W\right)}_{Q^L_{Z,e}} \; (e^-_L)^{[\text{weak}]}_j$$

$$\stackrel{\text{new basis}}{=} \overline{(e^-)}^{\text{mass}}_L \; E_L^{\dagger} \, \mathbb{1} \, E_R \; {}^{\text{mass}}$$

[similarly for $(e^-)_R$ and $(\nu_e)_L$ and photon coupling

— Finally, $\boxed{\text{Yukawa}}$ coupling in new basis is (5)
also diagonal (by construction), cf. in weak
basis, i.e.,

$$\mathcal{L}_{\text{Yukawa (lepton)}} \overset{\text{new basis}}{=} -\left(h_e^{\boxed{\text{diag.}}}\right)_{ij} \underbrace{\delta_{ij}}_{\text{note} \rightarrow \mathbb{1}} \bar{L}_i^{\text{mass}} \Phi\, e_{R_j}^{\text{mass}}$$

— Thus, charged lepton mass terms (from Φ VEV) are
$\boxed{\text{diagonal}}$ in new basis, hence/given the
name mass basis, i.e., $m_e = \left(h_e^{\text{diag.}}\right)_{11} v/\sqrt{2}$;
$m_\mu = \left(h_e^{\text{diag.}}\right)_{22}\, v/\sqrt{2}$ and $m_\tau = \left(h_e^{\text{diag.}}\right)_{33} \frac{v}{\sqrt{2}}$

— So are $\boxed{\text{couplings}}$ to Higgs bosons (physical & would-be NGB's)

— What about $\boxed{\text{neutrinos}}$ $\boxed{?!}$ Recall that in SM,
there are no mass terms for ν_L's (since we
don't add $\nu_{\boxed{R}}$'s). So, any basis for $\boxed{\nu_L\text{'s}}$
counts as "mass" basis (ν_L's are massless):
superscript "mass" on ν_L's, i.e., on entire
doublet, is then justified.

— Again, we will compare this next to quark
sector, where both components of $SU(2)_L$
doublet, i.e., up & down quarks, have mass terms

— In summary, SM lepton sector has a $\boxed{\text{global}}$
symmetry $\boxed{U(1)_e \times U(1)_\mu \times U(1)_\tau}$: $\boxed{\text{individual}}$
lepton number (i.e., for each generation) is
conserved

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- Just to be explicit, $\boxed{\mu^- \to e^- \gamma}$ is $\boxed{\text{not}}$ allowed in SM, but of course weak muon decay

$$\boxed{\mu^- \longrightarrow e^- + \nu_\mu + \bar{\nu}_e}$$

$U(1)_\mu$: $+1$ $\quad 0 \quad +1 \quad 0$

$U(1)_e$: $0 \quad 1 \quad 0 \quad \ominus 1$

[even if it involves μ^- to e^- "conversion"] is ok, since ν's carry lepton-number, thus preserving separately $U(1)_e$ & $U(1)_{\boxed{\mu}}$ [similarly, $U(1)_\tau$ for τ decays].

- Needless to say, $\boxed{m_\nu \neq 0}$ in beyond SM scenarios, so it is possible that $U(1)_{e,\mu,\tau}$ are separately broken (like happens in quark sector: see below).

- On to SM $\boxed{\text{quark}}$ sector, where flavor structure (i.e., type of up or down quark) is more rich

- The $\boxed{\text{idea}}$ is to simply follow the diagonalization procedure for Yukawa couplings, but now separately in down and up type quark sectors (each containing 3 generations), i.e.,

- $\boxed{\text{up}}$-quark sector: $h_u^{\boxed{\text{diag.}}} = U_L^\dagger \, h_u \, U_R$ ← general 3×3 matrix

so that $u_R^{(\text{mass})} = U_R^\dagger \, u_R^{\boxed{\text{weak}}}$ & $\boxed{u_L^{(\text{mass})} = U_L^\dagger \, u_L^{\boxed{\text{weak}}}}$

- (down)-quark sector: $h_d^{\boxed{\text{diag.}}} = D_L^\dagger \, h_d \, D_R$ ← different 3×3 matrix so that $d_R^{\text{mass}} = D_R^\dagger d_R^{\boxed{\text{weak}}}$ & $\boxed{d_L^{\text{mass}} = D_L^\dagger d_L^{\boxed{\text{weak}}}}$ (7)

— Here $U_{L,R}$ & $D_{L,R}$ are unitary transformations; in particular $\boxed{U_L \text{ \& } D_L}$ are not related, which will be crucial in what follows

— Just like for charged leptons, the new bases are labelled "mass" because couplings to $\boxed{\phi_0}$ (electrically neutral, but still complex part of Higgs field), including $\boxed{\text{VEV}}$ (v) and fluctuations around it: both radial (H) & angular (G), are diagonal (again, by construction):

$$\mathcal{L}_{\text{Yukawa, quark}} = -\overline{Q_i^{\boxed{\text{weak}}}}\left[h_u^{ij}\tilde{\Phi}\, u_{R_j}^{\boxed{\text{weak}}} + (h_d)_{ij}\,\Phi\, d_{R_j}^{\boxed{\text{weak}}}\right]$$

$$\ni -(h_u)_{ij}\left(\overline{u_L^{\boxed{\text{weak}}}}\right)_i \phi_0 \left(u_R^{\boxed{\text{weak}}}\right)_j - (h_d)^{ij}\, \overline{d_{L\,i}^{\boxed{\text{weak}}}}\, \phi_0\, d_{R\,j}^{\boxed{\text{weak}}}$$

$$= -h_u^{\boxed{\text{diag.}}}\, \overline{u_L^{\text{mass}}}\, \phi_0\, u_R^{\text{mass}} - h_d^{\boxed{\text{diag.}}}\, \overline{d_L^{\text{mass}}}\, \phi_0\, d_R^{\text{mass}}$$

— Obviously, in mass basis, all "neutral" current interactions (photon, Z & gluon) for both $\boxed{\text{up}}$ & $\boxed{\text{down}}$ quarks remain diagonal (in generation space), just like for $\boxed{\text{leptons}}$

— In particular, setting $\boxed{\phi_0}$ to just its $\boxed{\text{VEV}}$ gives quark masses

$$m_u = \left(h_u^{\boxed{\text{diag.}}}\right)_{11} v/\sqrt{2}\;;\; m_c = \left(h_u^{\boxed{\text{diag.}}}\right)_{22}\frac{v}{\sqrt{2}}\;;\; m_t = \left(h_u^{\boxed{\text{diag.}}}\right)_{33} \times v/\sqrt{2}$$

$$m_d = \left(h_d^{\boxed{\text{diag.}}}\right)_{11} v/\sqrt{2}\;;\; m_s = \left(h_d^{\boxed{\text{diag.}}}\right)_{22}\frac{v}{\sqrt{2}}\;;\; m_b = \left(h_d^{\boxed{\text{diag.}}}\right)_{33}\frac{v}{\sqrt{2}}$$

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- Remarkably, story for $\boxed{\text{charged}}$ current interactions, i.e, couplings to $W^\pm$ & $\phi_\pm$ (would-be NGB) is different than for neutral current (and for leptons) as follows
- The crucial point is that, in general, $\boxed{U_L \neq D_L}$, i.e, rotations to go from weak to mass basis are different for u_L vs $\boxed{d_L}$ (because $h_{u,d}$ are independent of each other)
- However, $u_L^{\boxed{\text{weak}}}$ & $d_L^{\boxed{\text{weak}}}$ make up $SU(2)_L$ doublet
- So, these rotations on u_L, d_L break $SU(2)_L$ gauge invariance, thus modifying $\boxed{W^\pm}$ couplings (from identity in $\boxed{\text{weak}}$ basis to not so in mass)

$$\mathcal{L}_{\substack{\text{quark, charged}\\ \text{current}}} = -\frac{g}{2^{1/2}} \overline{u_L^{\boxed{\text{weak}}}} \not{W}^+ \underbrace{\mathbb{1}}_{\substack{\text{in generation}\\ \text{space}\\ \text{(definition of weak basis)}}} d_L^{\boxed{\text{weak}}}$$

$$-(h_d)_{ij} \overline{u_{Li}^{\boxed{\text{weak}}}} \phi_+ d_{Rj}^{\boxed{\text{weak}}}$$

$$-(h_u)_{ij} \overline{d_{Li}^{\boxed{\text{weak}}}} \phi_- u_{Rj}^{\boxed{\text{weak}}} \quad (+\text{ h.c.})$$

new basis

$$= -\frac{g}{\sqrt{2}} \overline{u_L^{\text{mass}}} \underbrace{U_L^\dagger \mathbb{1} D_L}_{\equiv \boxed{V_{CKM}}} \not{W}^+ d_L^{\text{mass}} \quad (+\text{ h.c.})$$

$$-\phi_+ \left[\overline{u_L^{\text{mass}}} \boxed{V_{CKM}} h_d^{\boxed{\text{diag.}}} d_R^{\text{mass}} + \overline{u_R^{\text{mass}}} h_u^{\boxed{\text{diag.}}} \boxed{V_{CKM}} d_L^{\text{mass}} \right]$$

where 2nd, 3rd terms are $\boxed{\phi_\pm}$ couplings

- Thus, $\boxed{V_{CKM} \equiv U_L^\dagger D_L}$ is source of generational $\boxed{\text{mixing}}$ in SM quark sector, leading to observables of next note