BSM: (some) details of GUT

Reference: chapter 14 of Cheng, Li

- Outline (a) Motivation is aesthetic (at least to begin with), i.e., reduce number of "parameters" (including gauge quantum numbers of fermions) by simplifying structure.
- This is achieved by unifying 3 gauge groups/couplings of SM, i.e., $SU(3)_c$, $SU(2)_L$ & $U(1)_Y$, into single gauge group, e.g., $SU(5)$
- At the same time, quarks & leptons also can unify, i.e., fit nicely/completely into 2 representations ($\bar{5}$ and 10) of $SU(5)$ (per generation, resulting in quantization of their hypercharges (i.e, explanation of the seemingly "random" values in SM)...

... leading also to (b) interesting/testable phenomenology such as proton decay and unification of the (different) 3 low-energy gauge couplings when evolved to very high energies (similarly, possible unification of Yukawa couplings of quarks & leptons, which are of course different at low energies) [these meeting of couplings at high energies is of course a result of having less parameters in the underlying/high-energy theory]

- Onto $\overset{\text{some}}{\wedge}$ $\boxed{\text{details}}$: we <u>will focus</u> on simplest GUT, i.e., $\boxed{SU(5)}$ $\overset{\text{(gauged)}}{\wedge}$ broken down (spontaneously) to SM gauge group: $SU(3)_c \times SU(2)_L \times U(1)_Y$ (as we will discuss more below)
- In fundamental (5) representation, SU(5) generators ($\boxed{5\times 5}$ traceless, Hermitian matrices) contain the SM generators (also in $\overset{\text{respective}}{\wedge}$ fundamental representations) as follows:

$$\left[\begin{array}{c|c} \overset{\leftarrow \text{ non-zero}}{\lambda^{a=1\ldots 8}/2} & \\ 3\times 3 & 0_{3\times 2} \\ \text{(non-zero)} & \\ \hline 0_{2\times 3} & 0_{2\times 2} \end{array}\right] : \text{(usual) } \boxed{8} \text{ generators of } \boxed{SU(3)_c} \; ; \; \left[\begin{array}{c|c} 0_{3\times 3} & 0_{3\times 2} \\ \hline 0_{2\times 3} & \sigma^{a=1\ldots 3}/2 \end{array}\right] : \text{(usual) } \boxed{3} \text{ generators of } \boxed{SU(2)_L}$$

$\uparrow$ 2×2 (non-zero)

rank of SU(5) $\nwarrow$

and the 4th $\overset{\text{(purely)}}{\wedge}$ diagonal (traceless of course) generator [remaining 3 are 2 in $SU(3)_c$ & 1 in $SU(2)_L$] will be identified with $\boxed{\text{hypercharge}}$ (when we assingn SM fermion representations below): in $\boxed{SU(5)}$ normalization, i.e., trace $(T^a T^b) = \delta^{ab}/2$, it is given by

$\boxed{U(1)_Y}$ generator [all 5 rows/column contain $\boxed{1}$ non-zero entry, cf. $SU(3)_c$ or $SU(2)_L$ generators above]:

$$\hat{Y} = \sqrt{3/5} \begin{bmatrix} -1/3 & & & & \\ & -1/3 & & 0 & \\ & & -1/3 & & \\ & 0 & & +1/2 & \\ & & & & +1/2 \end{bmatrix} \quad \left\{ \text{again trace } \hat{Y}^2 = \frac{3}{5}\left[\left(\frac{1}{3}\right)^2 \cdot 3 + \left(-\frac{1}{2}\right)^2 \cdot 2\right] = 1/2 \right\}$$

$\uparrow$ SU(5) normalization

– Onto $\overset{\text{choice of}}{\wedge}$ SM fermion $\boxed{\text{representations}}$ under SU(5) so as to reproduce SM $\overset{\text{gauge}}{\wedge}$ quantum numbers:

for this purpose, it is convenient to "convert" $\boxed{R}$H SM fermions into Ⓛ H, i.e., re-write $(e^-)_R$ as $(e^c)_L$ ← conjugate, i.e., LH $\boxed{\text{positron}}$. Similarly, $(u^c)_L$ is LH anti-up quark and $(d^c)_L$ is LH anti-down quark.

{ Just to be clear, all of above – and the "originally" LH fermions, i.e., ν_L, $(e^-)_L$ [forming $SU(2)_L$ doublet] and $(u)_L$, $(d)_L$ (quark doublet) – are 2-component/ $\boxed{\text{Weyl}}$ spinors. So in Weyl/chiral basis for Dirac γ-matrices, the $\boxed{4-}$component/massive electron

$$\text{spinor} = \begin{bmatrix} (e^-)_L \\ (e^c)^*_L = (e^-)_R \end{bmatrix} \text{ etc.}$$ }

again, $Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$; $L = \begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix}$;

↑ $(e^-)_R$; u_R & d_R

– In this way, we see that all $\boxed{5}$ representations of SM $\boxed{\text{fermions}}$ (per generation) under SM gauge group, i.e., $SU(3)_c \times SU(2)_L \times U(1)_Y$, nicely fit inside $\overset{\text{(only)}}{\wedge}$ $\boxed{2}$ representations of $\boxed{SU(5)}$ as follows:

$\boxed{\text{(i).}}$ $\textcircled{\bar{5}}$ of SU(5) clearly contains $\left(\bar{3}, 1, +\frac{1}{3}\right)$, i.e., $\boxed{(d^c)_L}$

← $SU(3)_c$ ↙ $U(1)_Y$

↑ $SU(2)_L$ ↑ upto $\sqrt{\frac{3}{5}}$ from SU(5) normalization

and $(1, \bar{2}, -1/2)$, i.e., $\underset{\uparrow \text{ note}}{\textcircled{i\sigma_2}} L = \boxed{\begin{pmatrix} e_L \\ -\nu_L \end{pmatrix}}$

↑ recall $\bar{2}$ is equivalent to 2 for SU(2), i.e., "$\bar{2}$" $= i\sigma_2$ "2"

$$\bar{5} = \begin{bmatrix} (d^c)_L \\ (e^-)_L \\ -\nu_L \end{bmatrix}$$

$\boxed{\text{(ii).}}$ $\boxed{10}$ of $\boxed{SU(5)}$ is formed by anti-symmetric product of two $\boxed{5\text{'s}}$ of $SU(5)$ [d.o.f. matches, i.e, $10 = {}^5C_2$] $= \left[(3, 1, -1/3) + (1, 2, +1/2)\right]^{\text{"}2\text{"}}$, but antisymmetrize...

$$= \Big(\underbrace{3 \times 3|_{\text{antisymmetric}}}_{\bar{3}}, 1, \underbrace{-2/3}_{2 \times (-1/3)} \Big), \text{i.e.}, \boxed{(u^c)_L}$$

"based on" $3_\alpha 3_\beta 3_\gamma \epsilon^{\alpha\beta\gamma}$ being singlet

$$+ \Big(3, 2, \underbrace{-1/3 + 1/2}_{+1/6}\Big), \text{i.e.}, \boxed{Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}}$$

$$+ \Big(1, \underset{\hookrightarrow 2 \times 2 \text{ antisymmetric}}{1}, (1/2) \times 2\Big), \text{i.e.}, \boxed{(e^c)_L}$$

– Again, $\boxed{1}$ (full) generation of SM fermions $= \boxed{(\bar{5} + 10) \text{ of } SU(5)}$ with "explanation" of (quantization of) $\boxed{\text{hypercharges}}$ (upto $\sqrt{3/5}$ overall factor): within just $\boxed{SM}$, hypercharge values seem "arbitrary" (of course we choose them to fit electric charges), but with $SU(5)$, they must "fit" into $SU(5)$ representation, i.e., hypercharge is in some sense "non-abelian" also [like $SU(3)_c$ and $SU(2)_L$].

– Of course, $\boxed{2}$ immediate questions arise (whose answers are related);

$\boxed{\text{(a).}}$ $SU(5)$ "predicts" that the ③ gauge couplings of

SM should be the same ("unified"), but their observed values at low energies (and that is the key here!) are $\boxed{\text{different}}$

$\boxed{\text{(b).}}$ SU(5) has a total of $5^2-1=\boxed{24}$ generators, i.e, gauge bosons: $\boxed{\text{SM}}$ accounts for $\big(\underset{SU(3)_c}{8} + \underset{SU(2)_L}{3} + \underset{U(1)_Y}{1}\big)$ $=\boxed{12}$ out of these ...

... so we have $\boxed{12}$ "extra" / beyond SM gauge bosons (we'll figure out their SM quantum numbers soon): clearly these cannot be massless, since otherwise they would mediate long-range force between SM fermions (to who they are expected to couple).

- The $\boxed{\text{extra}}$ generators look like in 5 representation

$$\left[\begin{array}{c|c} 0_{3\times3} & \overset{\boxed{6}\ \text{complex elements here}}{\neq 0} \\ \hline \underset{\text{c.c. representation of upper, right elements}}{\neq 0} & 0_{2\times2} \end{array}\right]$$

- Now, adjoint of SU(5) – to which all gauge bosons belong can be obtained from

$$\bar{5}\times 5 = (\bar{3}, 1, +1/3)\text{“}\times\text{''}(3, 1, -1/3) \Big\} \text{ gives } (8\,[\text{from } (\bar{3}\times\bar{3})],\ 1,\ 0), \text{ i.e, gluons}$$

$$+ (1, \bar{2}, -1/2)\text{“}\times\text{''}(1, 2, +1/2) \Big\} \text{ gives } (1,\ 3\,[\text{from } (2\times\bar{2})],\ 0) \text{ i.e., } \boxed{W\text{'s}}$$

$$+ (1, 1, 0), \text{ i.e., hypercharge gauge boson from combination of above 2}$$

$$+ (\bar{3}, 1, +1/3)\text{“}\times\text{''}(1, 2, +1/2) + \text{h.c.} \Big\} \ \boxed{\text{extra}} \text{ gauge bosons: } \boxed{(\bar{3}, 2, +5/6) + \text{h.c.}}$$

– The extra gauge bosons are denoted by $\underbrace{(X, Y)}$ and their h.c. $(X^\dagger, Y^\dagger)$, where

form $SU(2)_L$ doublet with $\boxed{\text{each}}$ of X, Y being $\boxed{3}$ of $SU(3)_c$ (color index on X, Y not shown for simplicity)

– Again $\boxed{(X, Y) \sim (\bar{3}, 2, +5/6)}$, while

$\boxed{(X^\dagger, Y^\dagger) \sim (3, \bar{2}, -5/6)}$

– Let's schematically work out their $\boxed{\text{couplings}}$ to <u>SM fermions</u> in $\bar{5}$ and 10 representations of $SU(5)$:

$$\boxed{\bar{5}^\dagger \not{D} \bar{5}} \text{ contains } \boxed{[(d^c)_L]^\dagger (X, Y) (i\sigma_2 L)}$$

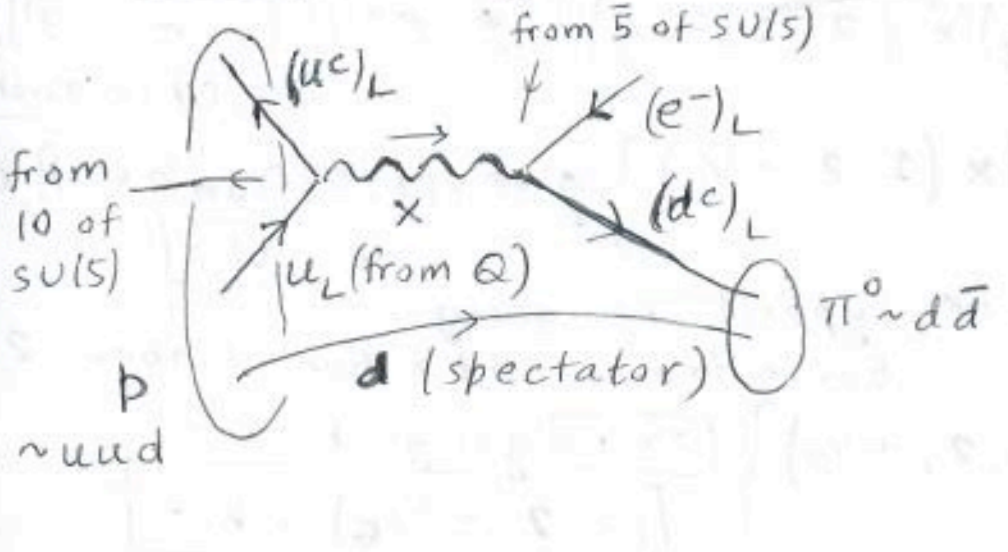
$$\left.\begin{array}{lccc} SU(3)_c: & 3 & \bar{3} & 1 \\ SU(2)_L: & 1 & 2 & \bar{2} \\ U(1)_Y: & -1/3 & +5/6 & -1/2 \end{array}\right\} \begin{array}{c}\text{check:}\\ \text{SM singlet}\end{array}$$

while $\boxed{10^\dagger \not{D}\, 10}$ gives $\boxed{Q^\dagger (X, Y) (u^c)_L}$

$$\left.\begin{array}{lccc} SU(3)_c & \bar{3} & \bar{3} & \bar{3} \\ SU(2)_L & \bar{2} & 2 & 1 \\ U(1)_Y & -1/6 & +5/6 & -2/3 \end{array}\right\} \begin{array}{c}\text{check:}\\ \text{SM singlet}\end{array}$$

and its h.c., i.e., $[(u^c)_L]^\dagger (X^\dagger, Y^\dagger)\, Q$

– Combining above 2 couplings gives $\boxed{\text{proton decay}}$ from $\boxed{(X, Y)}$ <u>exchange</u> (e.g., $p \to e^+ \pi^0$ as per below)!



... which (obviously) has $\boxed{\text{not}}$ been <u>seen</u> thus far, i.e., $\boxed{\text{proton}}$ <u>lifetime is</u> <u>very, very long</u>!

- So, clearly (X, Y) gauge bosons need to be super-heavy in order to suppress proton decay enough: we get effective (4-fermion) operator from X, Y exchange $\sim \frac{g^2_{SU(5)}}{M^2_{X,Y}} \underbrace{(e^-)_L}_{\text{creates } (e^+)_R} \underbrace{[(d^c)_L]^\dagger}_{\text{creates } \underline{\text{anti}}\text{-}d_R} u_L \underbrace{[(u^c)_L]^\dagger}_{\text{destroys } u_R}$

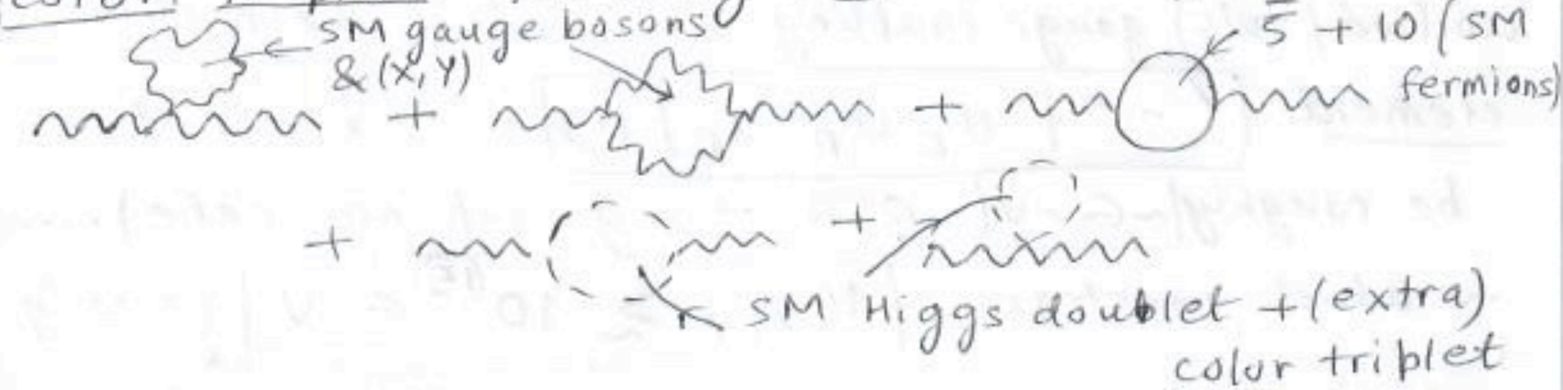
- We know $g_{SU(5)}$ [the unified/SU(5) gauge coupling: see below]; the hadronic matrix element $\boxed{\langle \pi^0 | \, u_L \, u_R \, d_R \, | p \rangle}$ is expected to be roughly $\boxed{\sim \text{GeV}}$ (QCD strong coupling scale)... ... which constrains $\boxed{M_{X,Y} \underset{\text{note: lower bound}}{\gtrsim} 10^{15} \text{ GeV}}$ (clearly at such energies, we can neglect Higgs VEV, i.e., assume W, Z are massless)

- So, the picture has to be 2 stages of SSB of gauge symmetries, i.e.,

$$SU(5) \xrightarrow[\boxed{X, Y}\text{ become massive}]{\gtrsim 10^{15}\text{ GeV}} SU(3)_c \times SU(2)_L \times U(1)_Y \xrightarrow[\sim 250 \text{ GeV}]{\text{(usual) Higgs VEV}}$$

$$\underbrace{SU(3)_c}_{\text{confines at } \sim \text{GeV}} \times U(1)_{EM} \; (W, Z \text{ massive})$$

- Clearly at energies (much) above $M_{X,Y} (\gtrsim 10^{15} \text{ GeV})$, SU(5) is a good symmetry. So, running of $SU(3)_c$, $SU(2)_L$ & $U(1)_Y$ gauge couplings is universal [indeed the 3 couplings are same, i.e, $g_{SU(5)}$]! This can be seen from complete SU(5) multiplets appearing in vacuum polarization diagrams: SM fermions form $(\bar{5}+10)$ per generation; SM gauge bosons combine with (X, Y) - which are approximately massless at these energies - to form 24 of SU(5) and SM Higgs doublet comes with a color triplet forming $\bar{5}_H$ of SU(5) (see below):



— However, at energies (much) below $M_{X,Y}$ (required by proton decay to be $\gtrsim 10^{15}$ GeV), X, Y gauge bosons "decouple" from vacuum polarization, while SM fermions continue to contribute universally (Higgs color triplet also decouples: see below).

— Thus, below energy scale $\sim M_{X,Y}$, the 3 gauge couplings evolve (run) differently [again because gauge bosons in loop are only SM, i.e., not a complete representation of SU(5)] ...

... which is what's needed to explain why the 3 gauge couplings are measured to be different at "low" energies, i.e., ~100 GeV (below which of course W's become massive)

- Actually, "flipping" above argument, we have a prediction that the ③ SM (low-energy) measured gauge couplings should "unify" when extrapolated (run up) to super-high energies $\boxed{\sim M_{X,Y}}$

- Indeed, very roughly, they do seem to meet (more precisely if we add superpartners) in this way.. ... that too around $\sim \boxed{10^{15}\ \text{GeV}}$, which is lower bound (again, roughly) on $\boxed{M_{X,Y}}$ from proton decay, i.e., there is a consistent picture here!

- Of course, as proton decay bounds get stronger, i.e., $M_{X,Y} \boxed{\gg} 10^{15}$ GeV, it will be difficult to accommodate this unification

- So, to summarize above discussion of gauge-fermion GUT sector:

(i). ③ SM gauge groups/couplings are unified into single group/coupling: $\boxed{SU(5)}$

(ii). Each fermion generation of SM fits into $\boxed{(\bar{5}+10)}$ representation of SU(5)

(iii). Above embedding "explains" (seemingly arbitrary) hypercharges of SM fermions

(iv). Extra (X, Y) gauge boson [i.e., in SU(5), but outside of SM] exchange gives proton decay $\Rightarrow \boxed{M_{X,Y} \gtrsim 10^{15}\ \text{GeV}}$, i.e., $\boxed{SU(5) \rightarrow SM}$ gauge group at super-high energies (see later for how to do this using scalar VEV).

(v). Below $M_{X,Y}$ (GUT scale), 3 SM gauge couplings run differently, thus (roughly) explaining their different low energy values

Next, we study fermion-scalar/Higgs sector: here's the outline for it:

(i). Minimally, SM Higgs doublet (Φ). is embedded in 5_H of SU(5), which also contains a color triplet (ϕ_c): this suffices to give both up & down-type quarks & leptons after EWSB as usual

(ii). Above structure gives unification of down-type quark and (charged) lepon Yukawa coupling ($h_d = h_e$). However, after SU(5) breaking/below $M_{X,Y}$ (GUT scale), $h_{d,e}$ run differently, roughly explaining $h_b > h_\tau$, i.e., for 3rd generation, but not for 2nd (where $h_s \sim h_\mu$)

(iii). Non-minimal embedding of SM Higgs doublet [i.e., into higher SU(5) representation] can give $h_s \sim h_\mu$

(iv). Color triplet exchange also gives (different) proton decay ⇒ $M_{\phi_c} \gtrsim 10^{11}$ GeV, vs. $M_\Phi \sim 10^{2}$ GeV, which can be "arranged" after SU(5) breaking

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→ Onto details of Yukawa couplings: the simplest possibility is to embed SM Higgs doublet, with hypercharge of $+1/2$ (as per our earlier convention), in a

5_H ← for Higgs of SU(5), i.e., $5_H = (3, 1, -1/3)$ Higgs color triplet (denoted by Φ_c ← colored)

$\oplus$ usual Higgs doublet (Φ) $\{(1, 2, +1/2)$

- The following SU(5)-invariant Yukawa couplings will suffice to give SM quark & (charged) lepton masses, with a relation between lepton & down-type quark masse

(a). $h_u [10\, 10\, 5_H] \ni h_u \underbrace{u^c}_{\text{from } 10} \underbrace{Q}_{\text{from } 10} \Phi$ ← from 5_H, i.e., up-quark Yukawa coupling [note that 10 is anti-symmetric product of two 5's so that above term is "product" of five 5's, which can be made SU(5) singlet using $\epsilon^{\alpha\beta\gamma\delta\rho}$]

$\boxed{(6).}$ $\overset{\text{see below}}{\left(h_{d(\text{or } e)}\right)}\left[\underbrace{\bar{5}\ 10}_{\text{SM fermions}}\ \underbrace{5_{\textcircled{H}}^{\dagger}}_{\text{h.c. of } 5_H}\right]$: note that

$\underline{\text{anti}}$-symmetric combination of $\bar{5}$ & $\overbrace{5_H^{\dagger}}^{\text{transforming as } \bar{5}}$ is $\overline{10}$, which forms SU(5) singlet with 10 in above Yukawa coupling

— As indicated above, this coupling gives (usual) charged $\boxed{\text{lepton}}$ Yukawa coupling $\sim \underbrace{i\sigma^2 L}_{\text{from } \bar{5}},\ \underbrace{(e^c)_L}_{\text{from } 10}\ \underbrace{\Phi^{\dagger}}_{\text{from } 5_H^{\dagger}}$ and that for $\boxed{\text{down}}$-type $\underline{\text{quark}} \sim \underbrace{(d^c)_L}_{\text{from } \bar{5}}\ \underbrace{Q}_{\text{from } 10}\ \overbrace{\Phi^{\dagger}}^{\text{from } 5_H^{\dagger}}$

— So, $\boxed{h_d = h_e}$ ... but $\boxed{\text{measured}}$ Yukawa couplings (rather charged lepton and down-type quark masses) are <u>not</u> equal ?!

... once again, SSB of GUT, coupled with running of couplings comes to the rescue here ($\boxed{\text{like}}$ for the 3 SM <u>gauge</u> couplings), i.e., $\boxed{h_d = h_e}$ is <u>valid</u> at $\boxed{\text{energies}}$ (much) <u>above</u> $\underline{M_{XY}}$ ($\gtrsim 10^{15}$ GeV), where SU(5) is a good symmetry ... whereas $\boxed{\text{non}}$-<u>equality</u> of <u>measured</u> $h_{d,e}$ is at <u>low energies</u>: the extrapolation between the two (running of Yukawa couplings) is non-universal, i.e, different for down-type quarks vs. leptons (analogous to 3 gauge couplings)

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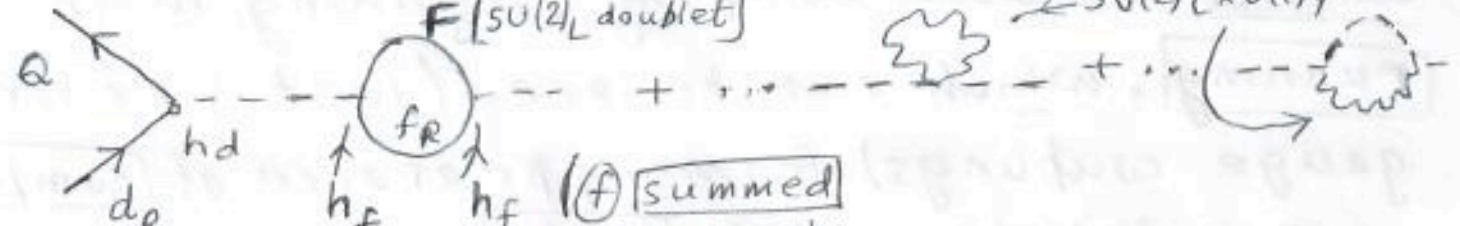
- Let's work out above-mentioned running of Yukawa couplings (including effects of GUT symmetry breaking), in part because it nicely illustrates difference from running of gauge couplings (e.g., absence of Ward-Takahashi identity with Yukawa couplings)
- Again, at energies (much) above $M_{X,Y}$ (scale of breaking of GUT symmetry), $h_d = h_e$, i.e., unified Yukawa coupling, including their running, which can be seen (just like for 3 gauge couplings) from appearance of complete GUT multiplets inside loops: we call this the "boundary condition" (BC) at energy $\sim M_{X,Y}$
- However, at energies below $M_{X,Y}$, even though SM fermions continue to appear in complete SU(5) multiplets, the bosonic multiplets are incomplete: only SM gauge bosons (no X, Y gauge bosons); similarly, only SM Higgs doublet [as we will see below, colored Higgs triplet (Φ_c) is "required" to be heavy and we will show how to achieve such a "doublet-triplet splitting"]
- for simplicity, here we take Φ_c to also have mass $\sim M_{X,Y}$

– Then, clearly running due to only SM in loop (vs. above $M_{X,Y}$) of h_d vs. h_e below $M_{X,Y}$ (or GUT scale) can be different [breaks SU(5) unifaction/universality]. So, even though we start with $h_d = h_e$ at $\boxed{M_{X,Y}}$, we get $\boxed{h_d \neq h_e}$ at $\boxed{\text{low}}$ energies.

– Let's consider $\boxed{\text{various}}$ contributions to $\boxed{\text{running}}$ of $\boxed{h_d}$ (again, below GUT scale, i.e., with only SM particles in loop): running of h_e can be similarly outlined, followed by a comparison of the 2 effects (again explaining $h_d \neq h_e$)

– $\boxed{\text{(I).}}$ $\boxed{\text{Wavefunction}}$ renormalization / self energy diagram of SM Higgs doublet ("Z_2" factor):



(f $\boxed{\text{summed}}$ over quarks & leptons)

i.e., involving $\boxed{\text{sum}}$ over (all) couplings of $\boxed{\text{Higgs}}$ doublet (gauge & Yukawa)

– clearly, this effect is "universal" for h_d vs h_e (or h_u for that matter), i.e., causes identical running ["analogous" to vacuum polarization / self energy of photon affecting (EM) coupling of electron and $\boxed{\text{quark}}$ in the same way]

– Note that $\boxed{\text{loop}}$ correction/running of (any) $h \propto h$ (bare) itself

i.e., $\boxed{\underbrace{\boxed{\delta h}}_{\text{loop correction}} \text{ (schematically)} \sim \boxed{h} \times (\text{possibly } \boxed{\text{other}} \text{ couplings})^2 / (16\pi^2)}$

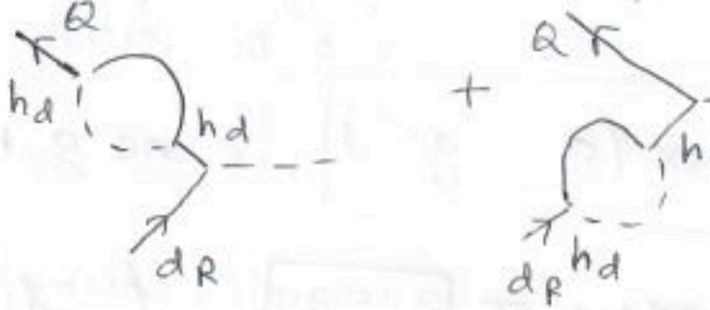
based on chiral symmetry, like for fermion mass (which in SM arises from Yukawa coupling and Higgs VEV of course), i.e. Yukawa coupling is between Ⓛ & $\boxed{R}$ chiralities of fermion...

- So, here we will mostly focus on $\boxed{\delta h/h} \sim \frac{(\text{possibly other couplings})^2}{16\pi^2}$,

i.e, by "identical" running, we mostly mean $\delta h/h$ is same for h_d vs. h_e etc.: $\left.\frac{\delta h_{d,e}}{h_{d,e}}\right|_{Z_{②} \text{ for (Higgs)}} \propto \boxed{\text{sum}}$ over $\left(\text{Higgs couplings}\right)^2$

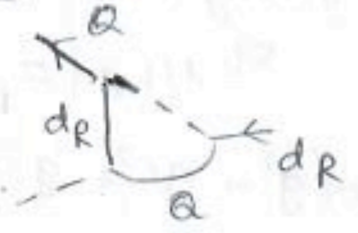
$\boxed{(II).}$ Corrections $\propto$ same $\boxed{\text{Yukawa}}$ coupling : 2 kinds

$\boxed{(a).}$ $\boxed{\text{wavefunction}}$ renormalization / self energy diagram of (fermions) (i.e., $\boxed{Z_2}$ for fermions, but Yukawa contribution):



, giving $\boxed{\left.\frac{\delta h_d}{h_d}\right|_{Z_{\boxed{2}}} \propto h_d^2}$ and $\frac{\delta h_e}{h_e} \propto h_e^2$

$\boxed{(b)}$. $\boxed{\text{Vertex}}$ correction (again $\propto$ same Yukawa coupling):



i.e., ("Z_1") factor: $\left.\frac{\delta h_{d,e}}{h_{d,e}}\right|_{Z_{①}} \propto h^2_{d,e}$

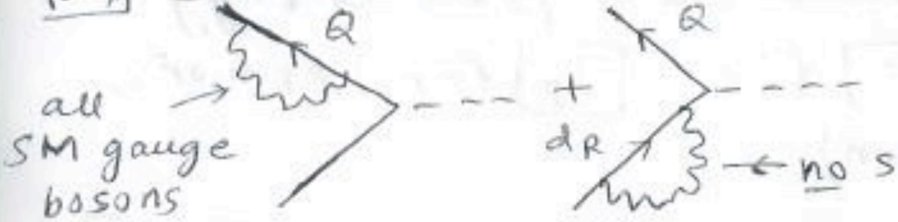
- Note that unlike for gauge coupling / theory, there is $\boxed{\text{no}}$ Ward-Takahashi identity so that above $Z_{1,2}$ effects do $\boxed{\text{not}}$ "cancel" each other, i.e.,

$$(\text{net}) \left.\frac{\delta h_{\underline{\underline{d,e}}}}{h_{\underline{\underline{d,e}}}}\right|_{(Z_1+Z_2)} \propto \underbrace{h^2_{\underline{d},\underline{e}}}_{\text{from } h_{d,e}} \left[\text{cf. } ⓢ Q/Q = 0 \text{ in QED due to } (Z_1 = Z_2) \text{ (cancellation)}\right]$$

→ The **net** correction "looks" non-universal (different for h_d vs. h_e).
However, it is clear that with **BC** given by $\boxed{h_d = h_e}$, running from just the above contribution (again Z_2 for fermion & Z_1, both from Yukawa coupling) will keep them equal

(III). Running due to gauge couplings of **fermions**:
again 2 kinds as for Yukawa couplings of fermions

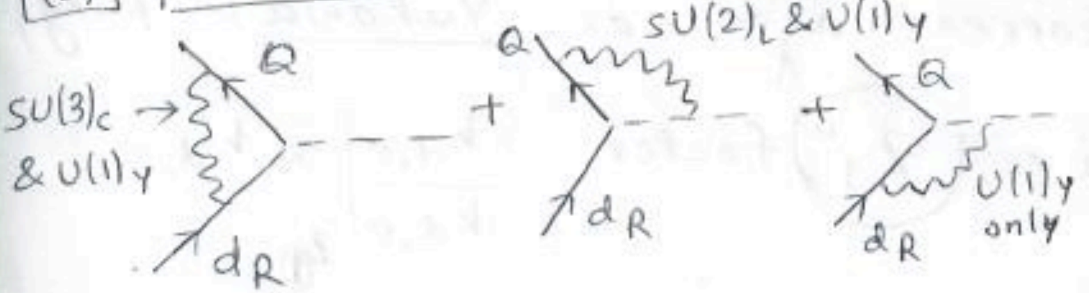
(a) **Wavefunction** renormalization of fermions (Z_2):



, giving $\delta h_d/h_d|_{Z_{\boxed{2}}} \propto (\underset{\text{QCD}}{g_s^2} + \underset{SU(2)_L}{g^2} + \underset{U(1)_Y}{g'^2})$

... clearly, $\boxed{\delta h_{\textcircled{e}}/h_e|_{Z_2} \propto (g^2 + g'^2)}$, i.e., no g_s here

(b) **Vertex** correction (again $\propto$ **gauge** coupling):



, giving (as above) $\delta h_d/h_d|_{Z_{\textcircled{1}}} \propto (g^2 + g'^2 + g_s^2)$

and $\delta h_{\textcircled{e}}/h_e|_{Z_1} \propto (g^2 + g'^2)$ (no g_s^2)

— Just like in (II) above, these $Z_{1,2}$ effects do not cancel [no Ward-Takahashi identity, cf. for gauge (EW) coupling of lepton vs. quark] so that

(net) $\boxed{\delta h_d/h_d|_{(Z_1+Z_2)} \text{ from gauge} \propto (g^2 + g'^2 + g_s^2)}$ and

$\delta h_e / h_e \Big|_{Z_1 + Z_2 \text{ from gauge}} \propto (g^2 + g'^2)$

i.e., non-universal (different for h_d vs. h_e), which is crucial here

— Clearly, due to effect in (III) above [i.e., vertex correction and fermion wavefunction renormalization from gauge coupling (again, no "g_s^2" for h_e, unlike for h_d : g'^2 also different due to values of Y)], at low energies we get $\boxed{h_d \neq h_e}$, even if $\boxed{h_d = h_e}$ at $\boxed{M_{X,Y}}$. In particular, SM fermion masses ~ (Higgs VEV) × h evaluated at weak scale, resulting in $m_d \neq m_e$.

— It turns out that gauge coupling effects in (III) tend to increase h in IR ⇒ we predict $\boxed{h_d > h_e}$ (again, h_e is "missing" g_s^2 that h_d has) at weak scale/low energies (if we start with $h_d = h_e$)

— Indeed, this works out for 3rd generation: $\boxed{m_b} \sim 4$ GeV (renormalized at ~GeV scale) $>$ $\boxed{m_\tau} \sim 1.7$ GeV

... but fails for 2nd generation: $m_s \sim 100$ MeV $\boxed{\sim}$ m_μ

— Solution for 2nd generation mass relation: SM Higgs doublet could arise (partly) from a higher SU(5) representation (instead of 5 assumed above). This will modify group theory structure of d vs. e Yukawa coupling, resulting in a different prediction/relation than $h_d = h_e$

(This "freedom" suggests that Yukawa coupling unification is somewhat "less robust" than that of gauge couplings.)

— $\boxed{\text{Back to}}$ minimal/simplest choice of SM Higgs doublet contained in $\boxed{5_H}$: recall that it comes with a $\boxed{\text{color}}$ triplet, $\Phi_{\boxed{c}}$; let's work out its couplings/phenomenology a bit

— We have $h_{d(or\,e)}\,\bar{5}\,10\,5_H^\dagger \ni h_d\,\underbrace{\overbrace{L\,Q}^{SU(2)_L\text{ singlet using }\epsilon^{ij}}\,\Phi_c^\dagger}_{SU(3)_c\text{ singlet}}$

$\ni h_d\,\underbrace{\overset{\text{Weyl spinors}}{e^-_L\,u_L}}\,\Phi_c^\dagger$

or $\overline{(e^c)}_R\,u_L$ in 4-component notation

$U(1)_Y: -1/2 + 1/6 - \underbrace{(-1/3)}_{Y\text{ of }SU(3)_c\text{ triplet in }5\text{ of }SU(5)} = 0$ (✓)

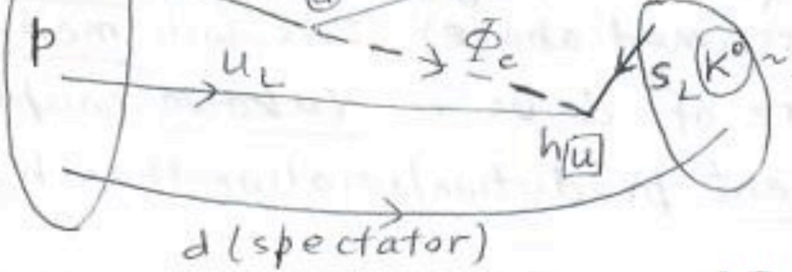
— Similarly, $h_u\,10\,10\,5_H \ni h_u\,\underbrace{\overbrace{Q_i\,Q_j}^{SU(2)_L\text{ singlet}}\,\Phi_c}_{SU(3)\text{ singlet using }\epsilon^{\alpha\beta\gamma}}$ (i: $\boxed{\text{generation}}$ index)

$\ni h_u\,\underbrace{\overset{\text{from }Q_{①}}{u_L}\,\overset{\text{from }Q_{\boxed{2}}}{ⓢ_L}}\,\Phi_c$

$U(1)_Y: +1/6 + 1/6 + (-1/3) = 0$ (✓)

Note: due to $\epsilon^{\alpha\beta\gamma}$ required to form $SU(3)_c$ singlet, the two Q's here cannot be from same generation

— Combining above $\boxed{2}$ couplings, we get (another) contribution to $\boxed{\text{proton decay}}$ from $\boxed{\Phi_c}$ exchange



$\sim \bar{s}d$, i.e., $\boxed{p \to K^0 e^+}$ [vs. $p \to \boxed{\pi^0} e^+$ from $\boxed{X, Y}$ exchange] with coefficient of 4-fermion operator, i.e., (schematically) $\sim \underbrace{(e^-)_L}_{\text{creates } e^+} \, u_L \, u_L \, \underbrace{s_L}_{\text{creates } \bar{s}}$ being

$$\sim \underset{\substack{\uparrow \\ 1^{st} \text{ generation}}}{h_d} \times \underset{\substack{\nwarrow \\ 2^{nd} \text{ generation}}}{h_c} \times \underset{\substack{\nwarrow \\ \text{1-2 mixing to get } u_L s_L \\ \text{starting with } \textcircled{c_L} s_L}}{\lambda_c} \times 1 / \boxed{M^2_{\Phi_c}}$$

[vs $\sim g^2_{SU(5)}/M^2_{X,Y}$ from heavy gauge boson exchange, which require $M_{X,Y} \gtrsim 10^{⑮}$ GeV]

- Since, $h_{d,c} \ll g_{SU(5)}$, lower bound on M_{Φ_c} from proton decay is "weaker" than for X, Y, i.e., (roughly) $\boxed{M_{\Phi_c} \gtrsim 10^{⑪} \text{ GeV}}$ ... which is still $\gg M_\Phi$ (SM Higgs doublet) $\sim 10^2$ GeV
- So, we need a solution to above Higgs $\boxed{\underset{\substack{\nwarrow \\ SU(2)_L}}{\text{doublet}} - \overset{\substack{SU(3)_c \\ \swarrow}}{\text{triplet}}}$ splitting problem, i.e., (again)

$$\boxed{\begin{array}{l} M_{\Phi_c} \, (\gtrsim 10^{11} \text{ GeV due to proton decay non-observation}) \\ \quad \gg M_\Phi \, (\sim 10^2 \text{ GeV for correct EWSB}) \end{array}}$$

when Φ, Φ_c are both part of 5 of SU(5), i.e., have same universal mass before $\boxed{SU(5)}$ breaking from

$$M^2_{5_H} 5_H^\dagger 5_H = M^2_{5H} \Big(\underbrace{\Phi^\dagger \Phi}_{\substack{\text{SM Higgs} \\ SU(2)_L \text{ doublet}}} + \underbrace{\Phi_c^\dagger \Phi_c}_{\substack{\text{color triplet} \\ \text{Higgs}}} \Big)$$

- Of course, GUT breaking effects can achieve this as follows: add $\boxed{\Sigma_{a=1..24}}$ scalar field transforming as adjoint of (gauged) SU(5), with following potential terms for 2 scalar fields:

$$\underbrace{V_\Sigma(\Sigma)}_{\text{only }\Sigma} + \underbrace{V_{5_H}(5_H)}_{\substack{\text{only } 5_H \\ \text{contains } M^2_{5_H} 5^\dagger_H 5_H \\ \text{of above}}} + \underbrace{V_{\Sigma,5_H}(5_H,\Sigma)}_{\substack{\text{couples }\Sigma\text{ to} \\ 5_H\text{, e.g.,}}}$$

$$\underbrace{\lambda \underbrace{\Sigma}_{\substack{\text{adjoint} \\ \text{of SU(5)}}} \underbrace{5^\dagger_H 5_H}_{\substack{\text{form }\boxed{\text{adjoint}} \\ \text{of SU(5),}}}}_{\text{form SU(5) singlet}}$$

– choose $V_\Sigma(\Sigma)$ so that its VEV breaks SU(5) down to $SU(3)_c \times SU(2)_L \times U(1)_Y$: (as usual)

writing $\boxed{\Sigma}$ as a $\boxed{5\times 5}$ matrix,

$$\langle \Sigma_a \underbrace{T^a}_{\substack{SU(5) \\ \text{generators} \\ \text{in fundamental} \\ \text{representation}}} \rangle \propto \left(\begin{array}{ccc|cc} 2 & & & & \\ & 2 & & & 0 \\ & & 2 & & \\ \hline & 0 & & -3 & \\ & & & & -3 \end{array}\right)$$

so that

$$\lambda \underbrace{5^\dagger_H}_{(\Phi_c^\dagger\ \Phi)\nearrow} \underbrace{\langle \Sigma \rangle}_{\substack{\text{above} \\ 5\times 5 \\ \text{matrix}}} \underbrace{5_H}_{\begin{pmatrix}\Phi_c \\ \Phi\end{pmatrix}} \sim \left[\enclose{circle}{2}\, \underbrace{\Phi_c^\dagger \Phi_c}_{\substack{\text{color} \\ \text{triplet}}} - \boxed{3}\, \overbrace{\Phi^\dagger \Phi}^{\text{SM Higgs doublet}} \right] \lambda \underbrace{v_\Sigma}_{\substack{\text{GUT} \\ \text{scale}}}$$

$$\text{– So, (net) } M^2_\Phi = M^2_{5_H} \enclose{circle}{-3} \overset{\substack{\text{dimensionful coupling} \\ (\sim \text{GUT scale})}}{\lambda} v_\Sigma$$

$$\text{vs. (net) } M^2_{\overset{\text{triplet}}{\Phi_c}} = M^2_{5_H} \boxed{+2} \lambda v_\Sigma$$

i.e., we can "cancel" $M^2_{5_H}$ with $3\lambda v_\Sigma$ so that M^2_Φ (SM Higgs $\boxed{\text{doublet}}$) $\sim (100\ \text{GeV})^2$ $\ll$ both $M^2_{5_H}$ & λv_Σ, each of which is $(\text{GUT scale})^2$

$\Rightarrow \enclose{circle}{M^2_{\overset{\text{triplet}}{\Phi_c}}}$ is clearly $(\text{GUT scale})^2$

[Rest of $V_{5_H}(5_H)$ contains usual quartic coupling for Φ]