the initially independent operators $a(t=0) \equiv a_0$ $b(t=0) = b_0$ become mixed as a result of their interaction.

The system commutator

$$[\hat a(t), \hat a^+(t)] = [\hat a_0, \hat a_0^+]\cos^2 \kappa t + [\hat b_0, \hat b_0^+]\sin^2 \kappa t$$
$$= 1.$$

it is preserved only by mixing in properties of the reservoir b-mode.

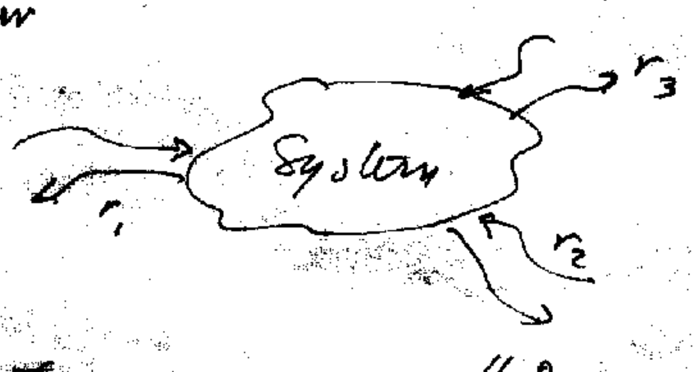
We need the operator character of the reservoir.

Jan, 29 2004

To obtain the irreversible evolution of the system (damping) one needs to consider a somewhat more complex model of the reservoir so that we get something other than the periodic exchange of excitation.

We need a continuum of degrees of freedom for the reservoir so that information can leave the system without returning (arbitrary long recurrence time).

Now



The system remains a H.O. but the reservoir is now taken to be a large set of H.O.

with creation / annihilation operators $\hat{r}_j^+$, $\hat{r}_j$
and independent couplings to the system.

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR}$$

$$\hat{H}_S = \hbar\omega_0 \hat{a}^+\hat{a} \qquad \hat{H}_R = \sum_j \hbar\omega_j \hat{r}_j^+\hat{r}_j$$

$$\hat{H}_{SR} = \sum_j \hbar\,(k_j \hat{a}^+\hat{r}_j + k_j^* \hat{a}\,\hat{r}_j^+)$$ again in the rotating wave approximation.

$$\hat{H}_{SR} = \hbar \quad \hat{a}^+\hat{\Gamma}^* + \hat{a}\hat{\Gamma}^+$$

where $\hat{\Gamma} = \sum_j k_j \hat{r}_j$

we want the Heisenberg equation of motion:

$$\dot{\hat{a}}(t) = \frac{1}{i\hbar}[\hat{a}, \hat{H}]$$

$$\dot{\hat{a}}(t) = -i\omega_0\,\hat{a}(t) - i\,\hat{\Gamma}(t)$$

and

$$\dot{\hat{r}}_j(t) = -i\omega_j \hat{r}_j(t) - ik_j^*\,\hat{a}(t)$$

together with the Hermitian conjugate equations.

One can already "hint" at a stochastic diff equation.
where the "drive force" comes from the reservoir $\hat{\Gamma}(t)$.
Looks like a Quantum Langevin equation.

There are two ways to proceed.

First we will attempt to eliminate the reservoir dynamics and will concentrate on the system dynamics to try to formulate a consistent description of the systems dynamics

The second approach is to stress the importance of the reservoir dynamics. That is precisely the output channel for information that is available to us as observers in the external world (input-output formalism). Championed recently by the French. E Giacobino, C. Fabre, S. Reynaud, A. Heidmann Phys Rev A 40, 1440 (1989)

Let us follow first approach and study the system's dynamics.

Start with

$$\dot{\hat{r}}_j(t) = -i\omega_j \hat{r}_j(t) - i k_j^* \hat{a}(t)$$

$$\underbrace{\dot{\hat{r}}_j(t) + i\omega_j \hat{r}_j(t)}_{\text{homogeneous equation}} = -i k_j \hat{a}(t)$$

Drive term.

Solution formally:

$$\hat{r}_j(t) = \hat{r}_j(0) e^{-i\omega_j t} - i \int_0^t dt' k_j^* \hat{a}(t') e^{-i\omega_j (t-t')}$$

inhomogeneous part.
Green's functions

If you have problems following that approach. let us solve it in an approximate way.

$$\dot{\hat{r}}_j(t) = -i\omega_j \hat{r}_j(t) - i k_j^* \hat{a}(t)$$

divide by $r_j(t)$

$$\frac{\dot{\hat{r}}_j(t)}{\hat{r}_j(t)} = -i\omega_j - i k_j^* \frac{\hat{a}(t)}{\hat{r}_j(t)} \quad \text{integrate.}$$

$$\ln \frac{\hat{r}_j(t)}{r_j(0)} = -i\omega_j t - i k_j^* \int_0^t \frac{a(t')}{r_j(t')} dt'$$

$$\hat{r}_j(t) = r_0 \exp[-i\omega_j t - i\int_0^t k_j^* \frac{a(t')}{r_j(t')} dt'$$

$$\hat{r}_j(t) = r_0(0) \exp[-i\omega_0 t] \exp[-i k_j^* \int_0^t \frac{a(t')}{r(t')} dt']$$

expand to first order in k

$$\hat{r}_j(t) = r_0(0) \exp[-i\omega t] [1 - i k_j^* \int_0^t \frac{a(t')}{r(t')} dt']$$

substitute the zeroth order solution for $r_j(t')$

$$r_j(t) = r_j(0) \exp[-i\omega_j t][1 - i k_j^* \int_0^t \frac{a(t') \, dt'}{r_j(0) \exp(-i\omega_j t')}$$

multiply.

$$r_j(t) = r_j(0) \exp[-i\omega_j t] - i k_j^* \int_0^t dt' a(t') \exp^{-i\omega_j(t-t')}$$

you can also check directly that the formal solution of the equation is exact by using Leibnitz rule to derive under integrals.

We substitute the formal solution into the equation for $\hat{a}(t)$

$$\dot{\hat{a}}(t) = -i\omega_0 \hat{a}(t) - \sum_j \int_0^t dt' |\kappa_j|^2 \hat{a}(t') e^{-i\omega_j (t-t')} \underbrace{- i\sum \kappa_j \hat{r}_j(0) e^{-i\omega_j t}}_{\hat{R}(t)}$$

Reservoir operator freely evolving.

This is just the formal solution to the problem, we need some approximations to make progress.
We will be quite sloppy with some mathematics.

Let us first look at an envelope.

$$\hat{a}(t) = \underbrace{\tilde{\hat{a}}(t)}_{\text{slow}} \underbrace{e^{-i\omega_0 t}}_{\text{fast}}$$

where $|\dot{\tilde{\hat{a}}}(t)| << \omega_0 \hat{a}(t)$

we measure $\tilde{\hat{a}}(t)$ and it occurs at a time scale Δt_d much larger than $\Delta t_{res} \approx \omega_0^{-1}$

Δt_{res} characterizes the evolution of the reservoir.

we will settle for a coarse grained description and will examine time intervals for which.

$\Delta t_{re} \ll \Delta t \ll \Delta t_d$.

Then we pass to the continuum limit by introducing the density of reservoir modes for the slowly varying amplitude $\tilde{a}$ its equation now reads:

comes from $\underline{t-t'}$

$$\dot{\tilde{a}}(t) = -\sum_j \int_0^t d\tau\, |k_j|^2 \tilde{a}(t-\tau) e^{-i(\omega_j-\omega_0)\tau}$$

$$- \sum_j k_j \hat{r}_j(0) e^{-i\omega t}$$

$$\approx -\int_0^\infty d\omega\, g(\omega) \int_0^t d\tau\, |k(\omega)|^2 \tilde{a}(t-\tau) e^{-i(\omega-\omega_0)\tau}$$

$$\underbrace{- \int_0^\infty d\omega\, g(\omega)\, k(\omega)\, \hat{r}_\omega(0)\, e^{-i\omega t} e^{i\omega_0 t}}_{i\, R(t)}$$

Because the exponential factor is rapidly varying averaging to zero

$\tilde{a}(t-\tau) \to \tilde{a}(t)$ except for $\tau = 0$.

(future depends on the present, but not on the past). (We will come back to this problem when we look at the master equation).

(Markovian approximation).

then:

$$\dot{\tilde{a}}(t) = -\left\{\int_0^\infty d\omega\, |\kappa(\omega)|^2 g(\omega)\, \xi(t,\omega)\right\} \tilde{a}(t) - i\, R(t)\, e^{i\omega_0 t}$$

with $\xi(t,\omega) \equiv \int_0^t d\tau\, e^{-i(\omega-\omega_0)\tau}$

since $\Delta t \gg \frac{1}{\omega_0}$ (coarse graining)

we take

$$\lim_{t\to\infty} \xi(t,\omega) \equiv \xi(\omega) = \pi\,\delta(\omega-\omega_0) + \frac{i\,P}{\omega_0-\omega}$$

then

$$\dot{\tilde{a}}(t) = -\alpha\,\tilde{a}(t) - i\,R(t)\,e^{i\omega_0 t}$$

with

$$\alpha \equiv \gamma + i\Delta$$

$$\gamma \equiv \pi\, g(\omega_0)\, |\kappa(\omega_0)|^2 \qquad \text{decay}$$

$$\Delta \equiv P\int_0^\infty d\omega\, \frac{g(\omega)\,|\kappa(\omega)|^2}{\omega_0-\omega} \qquad \text{frequency shift}$$

transforming back to the fast frame:

$$\omega_0' = \omega_0 + \Delta$$

$$\dot{\hat{a}}(t) = -(\gamma + i\omega_0')\,\hat{a}(t) - i\,\hat{R}(t)$$

the equation:

$$\dot{a}(t) = -(\gamma + i\omega_0)\,\hat{a}(t) - \underline{\underline{i\hat{R}(t)}}$$

noise process.

$$\Delta a = f(\gamma, \omega_0)\, a\, \Delta t + g(\alpha,\beta)\, dW$$

increment

Wiener Process.

This is an Euler's rule.
for integrating a differential equation.
at each time step dW is a Gaussian distributed random number with mean zero and variance Δt.
where Δt is the integration time step.

This ~~will~~ can ~~be~~ formally ~~be~~ be proven ~~by~~ from the Focker-Planck equation to Ito-Calculus.

C. W. Gardiner, Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences, Springer; Berlin 1983

This lends itself naturally to the Monte-Carlo simulation of the problem.

$$\dot{\hat{a}}(t) = -(\gamma + i\omega_0')\,\hat{a}(t) - i\hat{R}(t)$$

γ is the decay introduced by the interaction while Δ is the corresponding frequency shift.

(An electron in the Coulomb potential of the nucleus would give (γ, Δ) radiative decay rate and the frequency shift (Einstein A-coefficient and the Lamb shift). induced by the interaction with the reservoir formed by the quantized radiation fields.

There are questions about the mathematical justification of the steps taken in particular the convergence of the P∫ for Δ, except note that unlike the atomic case, all levels of the H.O. are shifted equally.

Now one can calculate and focus one's attention on the role of fluctuations of the reservoir on the system's dynamics.

Start with the mean value for $\langle \hat{a}(t) \rangle$.

For a reservoir state $\langle R \rangle = 0$

$$\langle \dot{\hat{a}}(t) \rangle = -(\gamma + i\omega_0')\langle \hat{a}(t) \rangle$$

just a classical oscillator. no "trace" of the reservoir except in γ and Δ.