Physics 711, Fall 2004, O.W. Greenberg

Solutions to delayed midterm exam

1. I'll do it in the simplest way, without using tables. Write the nucleon and pion multiplets as $|p\rangle = |1/2, 1/2\rangle,\ |n\rangle = |1/2, -1/2\rangle$ and
$|\pi^+\rangle = |1, 1\rangle,\ |\pi^0\rangle = |1, 0\rangle,\ |\pi_-\rangle = |1, -1\rangle$. Construct isospin generators for each of these multiplets so that the generators obey $[I_+, I_-] = 2I_3$, etc., *not* Georgi's normalization. Then, for the nucleon multiplet,
$I_+ = p^\dagger n,\ I_- = n^\dagger p,\ I_3 = (1/2)(p^\dagger p - n^\dagger n)$. For the pion multiplet, obviously
$I_3 = \pi^{+\dagger}\pi^+ - \pi^{-\dagger}\pi^-$. Try $I_+ = \alpha\pi^{+\dagger}\pi^+ + \beta\pi^{0\dagger}\pi^-$ and, of course, $I_- = I_+^\dagger$. Find α and β that make the generators obey the CR above. Can choose $\alpha = \beta = \sqrt{2}$.
Start with the highest weight in $|3/2, 3/2\rangle = |\pi^+ p\rangle$. Use
$I_- = I_-(\pi) \times 1 + 1 \times I_-(N)$ to lower the state; then normalize it. Keep doing this to generate all the $I = 3/2$ states. The results, including the $|3/2, 3/2\rangle$ which we already chose, are
$|3/2, 3/2\rangle = |\pi^+ p\rangle$,
$|3/2, 1/2\rangle = (1/\sqrt{3})[\sqrt{2}|\pi^0 p\rangle + |\pi^+ n\rangle]$,
$|3/2, -1/2\rangle = (1/\sqrt{3})[|\pi^- p\rangle + \sqrt{2}|\pi^0 n\rangle]$,
$|3/2, 3/2\rangle = |\pi^- n\rangle$.
Next construct $|1/2, 1/2\rangle$ to be orthogonal to $|3/2, 1/2\rangle$, and lower it with I_- to find
$|1/2, 1/2\rangle = (1/\sqrt{3})[-|\pi^0 p\rangle + \sqrt{2}|\pi^+ n\rangle]$,
$|1/2, -1/2\rangle = (1/\sqrt{3})[-\sqrt{2}|\pi^- p\rangle + |\pi^0 n\rangle]$.
Now solve for the various initial and final states,
$|\pi^+ p\rangle = |3/2, 3/2\rangle$,
$|\pi^0 p\rangle = (1/\sqrt{3})[\sqrt{2}|3/2, 1/2\rangle - |1/2, 1/2\rangle]$,
$|\pi^+ n\rangle = (1/\sqrt{3})[|3/2, 1/2\rangle + \sqrt{2}|1/2, 1/2\rangle]$,
$|\pi^- p\rangle = (1/\sqrt{3})[|3/2, -1/2\rangle - \sqrt{2}|1/2, -1/2\rangle]$,
$|\pi^0 n\rangle = (1/\sqrt{3})[\sqrt{2}|3/2, 1/2\rangle + |1/2, 1/2\rangle]$,
$|\pi^- n\rangle = |3/2, -3/2\rangle$.
The scattering amplitudes are independent of I_3 since the scattering conserves isospin, i.e., the transition operator is an isospin singlet. Then the amplitudes are
$\langle\pi^+ p|T|\pi^+ p\rangle = T(3/2)$,

$\langle\pi^0 p|T|\pi^0 p\rangle = (1/3)[2T(3/2) + T(1/2)]$,
$\langle\pi^0 p|T|\pi^+ n\rangle = (\sqrt{2}/3)[T(3/2) - T(1/2)]$,
$\langle\pi^- p|T|\pi^- p\rangle = (1/3)[T(3/2) + 2T(1/2)]$,
$\langle\pi^- p|T|\pi^0 n\rangle = (\sqrt{2}/3)[T(3/2) - T(1/2)]$.
$\langle\pi^- n|T|\pi^- n\rangle = T(3/2)$,
The cross sections are the absolute squares of these, ignoring phase space, which is about the same for all of them.

2. (a) With one quix, and some number of light quarks we have to find enough light quarks to make a 6^* which can couple with the quix to form a color singlet. All reps are color reps here. We have

$$3 \times 3 = 6 + 3^* = (2) + (1,1), \tag{1}$$

$$3 \times 3 \times 3 = 10 + 2(8) + 1 = (3) + 2(2,1) + (1,1,1), \tag{2}$$

$$3 \times 3 \times 3 \times 3 = 15_b + 3(15_a) + 2(6^*) + 3(1) = (4) + 3(3,1) + 2(2,2) + 3(2,1,1) \tag{3}$$

The notation here is that a Young pattern $(a, b, c, \cdots)$ has a boxes in the first row, b boxes in the second row, c boxes in the third row, etc. The first time we get a 6_c is in $qqqq$, so the simplest color-singlet state of a quix and quarks is $Qqqqq$.
We need the 6^* to bind with the quix to make a color singlet. To find the $SU(6)_{fs}$, that is flavor-spin, content that goes with an antisymmetric combination of quarks, we treat the quark as an $SU(18)$ object that reduces as

$$SU(18) \to SU(6)_{fs} \times S(3)_c; \;\; 18 \to (6, 3). \tag{4}$$

With a quark as an 18 we must find the antisymmetric four-quark states in the 3060, $(1,1,1,1)$ state of $SU(18)$. This takes a bit of work.

$$18 \times 18 = 171 + 153 = (2) + (1,1) \tag{5}$$

$$153 \times 18 = 1938 + 816 = (2,1) + (1,1,1) \tag{6}$$

$$816 \times 18 = 11628 + 3060 = (2,1,1) + (1,1,1,1) \tag{7}$$

Now we want to find the $SU(6)_{fs}$ reps that go with the antisymmetric 3060 of $SU(18)$. We use the reduction $SU(18) \to SU(6)_{fs} \times SU(3)_c$. We find

$$18 \to (6,3) = ((1),(1)) \tag{8}$$

$$153 \to (21, 3^*) = ((2), (1,1)) \tag{9}$$

$$816 \to (70,8)+(56,1)+(20,10) = ((2,1),(2,1))+((3),(1,1,1))+((1,1,1),(3)) \tag{10}$$

$$\begin{aligned} 3060 = (210,3) + (105,6^*) + (105,15_a) + (15,15_b) = \\ ((3,1),(2,1,1)) + ((2,2),(2,2)) + ((2,1,1),(3,1)) + ((1,1,1,1),(4)) \end{aligned} \tag{11}$$

Since we want the 6^* for the color, we want the 105 for the flavor and spin.
We have to reduce $SU(6)_{fs} \to SU(3)_f \times SU(2)_s$. We find

$$105 \to (15_b, 1) + (15_a, 3) + (6^*, 5) + (6^*, 1) + (3, 3). \tag{12}$$

We should now decompose the $SU(3)_f$ multiplets into $SU(2)_I, Y$ multiplets. I won't do this in detail, but just note that since the $SU(2)_s$ reps $5, 3, 1$ occur there will be particles of spins $2, 1, 0$. The 15_b of $SU(3)_{fs}$ will have $(2I+1, Y)$ reps from $(1, -7/3)$ up to $(5, 5/3)$.

(b) The simplest color-singlet state of a quix and antiquarks is $Q\bar{q}\bar{q}$. We need the antisymmetric part,

$$(18^* \times 18^*)_{anti} = 153^* \tag{13}$$

We decompose

$$153^* \to (21^*, 3^*) + (15^*, 6^*) \tag{14}$$

Since we want the 6^* of color, we need the 15^* of flavor-spin.

$$15^* \to (6^*, 1) + (3, 3) \tag{15}$$

under $SU(6)_{fs} \to SU(3)_f \times SU(2)_s$. The 6^* has spin 0 particles with the quantum nos. of symmetric combinations of $\bar{d}\bar{d}, \bar{d}\bar{u}, \bar{u}\bar{u}$, with isospin 1, $\bar{d}\bar{s}, \bar{u}\bar{s}$, with isospin 1/2, and $\bar{s}\bar{s}$ with isospin 0. The 3 has particles of spin 1 with the quantum nos. of antisymmetric combinations of $\bar{d}\bar{u}$ of isospin 0 and $\bar{d}\bar{s}, \bar{u}\bar{s}$ of isospin 1.

(c) In color,

$$3 \times 3 = 6 + 3^* \tag{16}$$

$$3 \times 3 \times 3^* = 15_a + 6^* + 2(3) \tag{17}$$

We want the 6^*.

Next we need $SU(18)$ reps,

$$(18 \times 18)_{anti} = 153 = (1, 1) \tag{18}$$

$$(18 \times 18)_{anti} \times 18^* = 2736 + 18 = (18,1) + (17,2) \tag{19}$$

where here the notation refers to columns rather than rows. Next we decompose into $SU(6)_{fs)} \times SU(3)_c$.

$$(18 \times 18)_{anti} \times 18^* \to (120 + 6, 6^* + 3) + (84 + 6, 15_a + 3) \tag{20}$$

We want the 6^* of color, so the $SU(6)_{fs}$ reps are 6 and 120. I will forgo their decomposition into $SU(3)_f \times SU(s)_s$, etc.

(d) The decays depend on details of the interactions. The simplest case is where a particle can just fall apart; i.e. where the state contains an ordinary baryon with the correct color structure. For a baryon that would require having $(1,1,1)$ as an antisymmetric color singlet. The only state with three or more quarks is the $Qqqqq$, but this has the quarks in $(2,2)$, so it can't decay. Thus none of these states can decay to baryons. The state $Qq\bar{q}$ could contain the $q\bar{q}$ in a color singlet, but then the remaining state would be Qq which is not a color singlet. So none of the states can fall apart into color singlets.

3. (a) $[a_2, \Lambda_2][a_1, \Lambda_1] = [a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1]$.

(b) $[0, 1]$.

(c) $[-\Lambda^{-1}a, \Lambda^{-1}]$.

(d) The set associated by conjugation with $[0, \Lambda]$ is $[a_1 - \Lambda_1 \Lambda \Lambda_1^{-1} a_1, \Lambda_1 \Lambda \Lambda_1^{-1}]$ which is not in the subgroup $\{[0, \Lambda]\}$, so $[0, \Lambda]$ is not an invariant subgroup. The set associated with $[a, 1]$ is $[a_1, \Lambda_1][a, 1][a_1, \Lambda]^{-1} = [\Lambda_1 a, 1]$ which is in the subgroup $\{[a, 1]\}$, so $[a, 1]$ is an invariant subgroup, since the left and right cosets are the same.

e. We map the coset $[a_1, \Lambda_1][a, 1] = [a_1 + \Lambda_1 a, \Lambda_1]$ to the element Λ_1, so $SO(n,1)/[a,1] \cong SO(n)$

4. Since the $SU(3)_c$ has to be antisymmetric to be a singlet, we can take the three quarks in an s-state to be in a symmetric state in $SU(6)_{fs}$. So we need

$$6 \times 6 = 21 + 15 = (2) + (11) \tag{21}$$

$$15 \times 5 = 70 + 56 = (2,1) + (3) \tag{22}$$

We need the symmetric 56 which decomposes under $SU(6)_{fs} \to SU(3)_f \times SU(2)_s$ as

$$6 \to (3, 2) \tag{23}$$

$$(3, 2) \times (3, 2) \to [(6, 3) + (3^*, 1)]_{sym} + [(6, 1) + (3^*, 3)]_{anti} \tag{24}$$

$$[(6, 3)+(3^*, 1)]_{sym} \times (3, 2) \to [(10, 4)+(8, 2)]_{sym} + [10, 2)+(8, 4)+(8, 2)+(1, 2)]_{mixed} \tag{25}$$

Thus

$$56 \to [(10, 4) + (8, 2)] \tag{26}$$

The states in $(10, 4)$ are Δ, Y_1, Ξ, Ω all with spin 3/2. The states in $(8, 2)$ are N, Σ, Λ, Ξ all with spin 1/2.

The states with one particle in a P-state are in the 70 (see Eq.(22 above), which decomposes to

$$70 \to [10, 2) + (8, 4) + (8, 2) + (1, 2)]_{mixed} \tag{27}$$

Since the 70 has one quark in a p-state, we have to couple the $L = 1$ orbital angular momentum to the spin angular momenta of the quarks. This give more states, all negative parity. Altogether there should be $70 \times 3 = 210$ states connected with the 70. We list all the states, first those in the sss configuration and then those in the ssp configuration:

J^P	(D, L^P)	S	*Octet or Decuplet*	*Singlet*
$1/2^+$	$(56, 0^+)$	$1/2$	*Octet*	
$3/2^+$	$(56, 0^+)$	$3/2$	*Decuplet*	
$1/2^-$	$(70, 1^-)$	$1/2$	*Octet*	*Singlet*
$3/2^-$	$(70, 1^-)$	$1/2$	*Octet*	*Singlet*
$1/2^-$	$(70, 1^-)$	$3/2$	*Octet*	
$3/2^-$	$(70, 1^-)$	$3/2$	*Octet*	
$5/2^-$	$(70, 1^-)$	$3/2$	*Octet*	
$1/2^-$	$(70, 1^-)$	$1/2$	*Decuplet*	
$3/2^-$	$(70, 1^-)$	$1/2$	*Decuplet*	

In Manley's Table 1, the rows 1,3,4,5,6, and 12 correspond to rows 1,3,4,5,6,8 above; in addition the state $N(1700)3/2^-$ mentioned in Manley on p231, line 3 probably corresponds to the nucleon in row 7 in my table above. These states were suggested in my paper in 1964, more than 40 years ago.

As stated in the hint, the states *ssp* in the 56 correspond to center-of-mass motion and are not excitations of the ground state 56.