Physics 711, Symmetry Problems in Physics Fall 2005

Homework

Solutions for assignment due 9/13/05

Georgi 1.A. Always have a cyclic group Z_n with n elements, so have Z_3 with elements e, a, a^2, $a^3 = e$.
Uniqueness: every element of a finite group has a period of some order. The period forms a subgroup whose order must divide the order of the group. Since 3 is prime, the only period is the group itself, so Z_3 is unique. Like all Z_n it is abelian.

1.B. There is an element of order 4, so have Z_4 with elements e, a, a^2, a^3 $a^4 = e$.
There is a period of order 2, so

Group table of Z_4

	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

unique

Group table of period of order 2

	e	a	b	c
e	e	a	b	c
a	a	e		
b	b			
c	c			

$\longrightarrow$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

no element can repeat

If $b^2 = a$, then $b^4 = e$ and we get a rearrangement of Z_4. So need $b^2 = e$. The rest follows.

This is the "four group." Matrix rep is

$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $a = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.   Abelian.

1.C. Take the defining rep of S_n to act on the basis vectors $|i\rangle$, $i = 1, 2, \cdots, n$. Any perm is a product of transpositions, so suffices to find a (proper) invariant subspace under an arbitrary transposition (ij).

$$D((ij)) = |i\rangle\langle j| + |j\rangle\langle i| + \sum_{k\neq i,j} |k\rangle\langle k|. \tag{1}$$

Guess that $\sum_l |l\rangle$ is invariant. Check:

$$D((ij)) = \sum_l \delta_{jl}|i\rangle + \delta_{il}|j\rangle + \sum_{k\neq i,j} |k\rangle\delta_{lk} = \sum_l |l\rangle \tag{2}$$

Identity is unique: If $ea = a$, for all a and $e_1 = a$, for all a then $ea = e_1 a$ for all a so mult by a^{-1} from the right to get $e = e_1$.

Inverse is unique: If $xa = e$ and $x_1 a = e$, then $xa = x_1 a$. Mult by a^{-1} from then right to get $x = x_1$, so x is the unique inverse of a.

If a right identity and a right inverse exist, then a left identity and a left identity exist: Assume $ax = e$ (right inverse for a), $xb = e$ (right inverse for x), $ae = e$, (right identity). Then for left inverse
$xa = (xa)e = (xa)(xb) = x[(ax)b] = x(eb) = (xe)b = xb = e$, so x is also a left inverse to a.
For left identity
$a = ae = a(xa)$ (we now know that $xa = e$) $= (ax)a = ea$, so e is also a left identity.