

Lecture 21: Rotation for spin-1/2 particle, Wednesday, Oct. 26

Representations

$SO(3)$ is a group of three dimensional rotations, consisting of 3 rotation matrices $R(\vec{\theta})$, with multiplication defined as the usual matrix multiplication.

For a quantum mechanical system, every rotation of the system generates a new state $|\psi'\rangle$, which must be a unitary transformation of the state

$$|\psi'\rangle = \hat{U}(R)|\psi\rangle \quad (6)$$

because the probability must be conserved after rotation. Furthermore, when R is identity matrix, $U = 1$. When there are two successive rotations $R_2 R_1 = R_3$, the corresponding unitary matrices satisfy,

$$U(R_2)U(R_1) = U(R_2 R_1) = U(R_3) \quad (7)$$

A set of unitary matrices $U(R)$ which have the same group structure as the rotational matrices is called a representation of $SO(3)$.

The dimension of $U(R)$ can be finite and can also be infinite dimensional. One of the goal of the group theory is to find all possible representations of a group.

In most cases, the R-to- $U(R)$ correspondence is one-to-one. However, in certain cases, it is not. Not all representations are unitary. In particular, non-compact groups have non-unitary representations. For compact groups, all representations are unitary.

For infinitesimal rotations, we can write

$$U(R) = 1 - \frac{i}{\hbar} \vec{J} \cdot \vec{\theta} \quad (8)$$

where $\vec{J}$ are the generators of rotations in the state space. From comparison with the conservation laws in classical mechanics, we identify $\vec{J}$ as *quantum angular momentum operator*. It satisfies the same commutation relations as the generators of $SO(3)$,

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \quad (9)$$

A finite rotation can be constructed from the infinitesimal rotation as

$$U(R) = \exp\left(-\frac{i}{\hbar} \vec{J} \cdot \vec{\theta}\right) \quad (10)$$

Therefore, to find a representation of the group is equivalent to find a representation of J_i satisfying the right commutation relations, or the representation of the algebra.

2D representation

The simplest representation of $SO(3)$ algebra is 2D, with

$$J_i = \frac{\hbar}{2} \sigma_i \quad (11)$$

where σ_i are Pauli matrices. Some useful properties of Pauli matrices,

$$\begin{aligned} \{\sigma_i, \sigma_j\} &= 2\delta_{ij} \\ [\sigma_i, \sigma_j] &= 2i\epsilon_{ijk}\sigma_k \\ \text{Tr}\sigma_i &= 0 \\ \sigma_i^\dagger &= \sigma_i \\ \text{Det}\sigma_i &= -1 \\ (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) &= \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b}) \\ (\vec{\sigma} \cdot \vec{a})^2 &= |\vec{a}|^2 \end{aligned} \quad (12)$$

And the corresponding unitary matrices are

$$\begin{aligned} U(\vec{\theta}) &= \exp\left(-i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \\ &= \cos \frac{\theta}{2} - i\vec{\sigma} \cdot \hat{\theta} \sin \frac{\theta}{2} \\ &= \begin{pmatrix} \cos \frac{\theta}{2} - i\hat{\theta}_z \sin \frac{\theta}{2} & -(i\hat{\theta}_x + \hat{\theta}_y) \sin \frac{\theta}{2} \\ -(i\hat{\theta}_x - \hat{\theta}_y) \sin \frac{\theta}{2} & \cos \frac{\theta}{2} + i\hat{\theta}_z \sin \frac{\theta}{2} \end{pmatrix} \end{aligned} \quad (13)$$

It is interesting to see that U does not return to itself after 2π rotation. Rather, it becomes $-U$. For this reason, the representation is not one-to-one, it is one-to-two.

Any two dimensional vector which transforms under spatial rotations in terms of the above unitary matrix is called spinor. Let χ is a spinor one has,

$$\chi' = \exp(-i\vec{\sigma} \cdot \vec{\theta}/2)\chi \quad (14)$$

Construct the eigenstate of $\vec{n} \cdot \vec{S}$, as an exercise of rotation. We start with eigenstate of S_z . We can get to the (θ, ϕ) direction by first rotating in the

y direction by angle θ and then in the z direction by angle ϕ . The resulting state is

$$\chi = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \quad (15)$$

The physics of the rotating state can also be seen as follows. Suppose the spinor of the system is $|\psi\rangle = \text{column}(\alpha, \beta)$. Let us make a rotation of the system along the z-direction. Therefore, we have

$$|\psi'\rangle = \exp(-iS_z\phi/\hbar)|\psi\rangle \quad (16)$$

Let us now calculate the expectation value of S_x in the new state, one finds

$$\langle S_x \rangle' = \langle S_x \rangle \cos \phi - \langle S_y \rangle \sin \phi \quad (17)$$

This is in fact, true for any vector operators.

Unitary group SU(2)

I will not discuss this in the class. Consider a 2×2 unitary matrix which depends on four complex numbers or 8 real numbers

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (18)$$

The unitarity condition $U^\dagger U = U U^\dagger = 1$ imposes four conditions. Therefore, a general unitary matrix depends on four real numbers. The determinant satisfies the condition

$$|\text{Det}U| = 1, \quad \text{Det}U = \exp(i\alpha) \quad (19)$$

If we restrict that $\text{Det}U = 1$, we call U a special unitary matrix.

The generators of the SU(2) group can be defined to satisfy the same commutation relations as those of SO(3). Therefore, we say SU(2) and SO(3) groups have the same Lie algebra. The group manifold of SU(2) is twice as large as SO(3), is called the covering group of SO(3).