Lecture $\boxed{9}$ (Sept. 21, Mon.)

Outline for today (HW2 due, HW 3,4 assigned)

- continue dynamics of QM (time-evolution of states) in more general / mathematical framework (still not using usual Schroedinger wave equation for wavefunction: lot can be done without it!
- another example of $\boxed{2}$-state system time-evolution: neutrino (ν) oscillations
- Schroedinger picture so far (state evolves in time; operators do not) vs. Heisenberg picture (states time-independent; operators evolve...)

$\boxed{\text{Neutrino}}$ oscillations briefly (HW 4.1 for details)

(electrically neutral, very light, weakly interacting particles)

- 2 "flavors" (simplified version): $\nu_{e,\mu\,(\text{muon})}$ interacting only with e, μ
- Energy/mass eigenstates are linear combinations of flavor eigenstates (and vice versa)

$$|\nu_{e,\mu}\rangle = \cos\theta\,/\,\sin\theta\,|\nu_1\rangle - \sin\theta\,/\,(+\cos\theta)\,|\nu_2\rangle$$

with $m_1 \neq m_2$

- $\boxed{t=0}$: $|\nu_e\rangle$ produced using e^- (source), but its two mass eigenstate "components" evolve in time with different "frequencies" $\Rightarrow$ different linear combination at later time t: finite probability to find $|\nu_\mu\rangle$ (produces μ in detector), probability to detect as $|\nu_e\rangle < 1$ (like S_y+ initially, detected as S_y- later)

Summary of 2-state system time evolution

(spin precession, neutrino oscillations)

- initial state is linear combination (l.c.) of (2) energy eigenstates $\Rightarrow$ (2) components evolve with different frequencies

$\Rightarrow$ at later time, a different l.c. (state evolves non-trivially)

Schroedinger $^{(S)}$ vs. Heisenberg $^{(H)}$ pictures

So far (more familiar/usual picture: called S), state ket evolves in time, but operators do not change ... vs. H-picture

(Motivation: different viewpoint can lead to better, simpler understanding of same result)

— predictions intact ($\boxed{\text{inner products}}$)

$$\underbrace{\langle \beta | \alpha \rangle}_{\text{initial}} \xrightarrow[\text{time}, t]{\text{later}} (\langle \beta | U^\dagger)\underbrace{(U|\alpha\rangle)}_{\text{new ket}} = \langle \beta | \alpha \rangle \text{ (trivial)}$$

$$\langle \beta | X | \alpha \rangle \longrightarrow (\langle \beta | U^\dagger) X (U|\alpha\rangle) \quad \ldots (1) \text{ S-picture}$$

$$= \langle \beta | (U^\dagger X U) | \alpha \rangle \quad \ldots (2) \text{ H-picture}$$

(not constant, unless $[X, H] = 0$)

$U = \exp\left(\frac{-iHt}{\hbar}\right)$

(1) S-picture: X unchanged, but $|\alpha\rangle, |\beta\rangle$ evolve

(2) H-picture: $X \rightarrow U^\dagger X U$ (evolves), with $|\alpha\rangle\, |\beta\rangle$ unchanged

$$\boxed{A^{(H)}(t) = U^\dagger(t)\, A^{(S)}\, U(t)}$$

($A^{(H,S)}$ are unitary equivalent operators)

$\boxed{t = 0}$ both pictures same

$|\alpha, t_0 = 0\rangle$ S or H (drop) & $A^{(H)}(t=0) = A^{(S)}$

later time, t $\qquad A^{(H)}(t) \neq A^{(S)}$ / unless $[A^{(S)}, H] = 0$

in that case, $A^{(H)} = A^{(S)} =$ constant

& $|\alpha, t_0 = 0, t\rangle_H = |\alpha, t_0 = 0\rangle$

$\neq |\alpha, t_0 = 0, t\rangle_S = U(t)|\alpha, t_0 = 0\rangle$

but ${}_S\langle \alpha, t_0 = 0, t| A^{(S)} |\alpha, t_0 = 0, t\rangle_S = \ldots H \ldots$

(by "construction"!)

H - EOM : [analog of $i\hbar\, \partial/\partial t\, |\alpha, t_0; t\rangle_S = H|\alpha, t_0; t\rangle_S$ (of S-picture)]

$$dA^{(H)}/dt = \partial U^\dagger/\partial t\, A^{(S)} U + U^\dagger A^{(S)} \partial U/\partial t$$

($A^{(S)}$ time-independent)

[use : $\boxed{U = \exp(-iHt/\hbar)}$: H is time-independent]

$$\ldots \boxed{dA^{(H)}/dt = \frac{1}{i\hbar}[A^{(H)}, H]}$$

Sanity checks : (a) If $[A^{(H)}, H] = 0 \Rightarrow$ use & $U = \exp(-iHt/\hbar)$

$A^{(S)} = U A^{(H)} U^\dagger = A^{(H)}$ [knew already that $A^{(H)}$ constant in this case]

$=$ constant)

(b) ${}_S\langle a'| X^{(S)} \underbrace{|a'\rangle_S}_{\text{stationary state}} =$ constant (earlier) $\leftarrow$ general

stationary state ($[A, H] = 0$)

In H-picture $\frac{d}{dt}\, {}_H\langle a'| X^{(H)} |a'\rangle_H = {}_H\langle a'| X^{(H)} H - H X^{(H)} |a'\rangle_H$

act here only

$E_{a'}$ $\qquad E_{a'}$

$= 0$ (agrees...)

– $\boxed{\text{constructing } H}$: In classical H, replace x, p by operators $\left[x\,p \longrightarrow \frac{1}{2}(xp + px) \right]$ (prescription)

– Useful formula : $[x, F(p)] = i\hbar\, \partial F/\partial p$;

$$[p, G(x)] = -i\hbar\, \partial G/\partial x$$

$\boxed{\text{Free}}$ particle : $H = p^2/(2m) \Rightarrow \frac{dp}{dt} = \frac{1}{i\hbar}\underbrace{[p, H(p)]}_{=0}$

$\boxed{p(t) = p(0)}$ [as expected from general result]

$$dx/dt = \left[x, \frac{p^2}{2m}\right]\frac{1}{i\hbar} = p(t)/m = p(0)/m$$

$$\Rightarrow \boxed{x(t) = x(0) + p(0)/m\; t}$$

plug

$$\Rightarrow [x(t), x(0)] = -i\hbar/m\; t \neq 0$$

$\Rightarrow$ uncertainty relation : (use general formula) $\boxed{\langle(\Delta x)^2\rangle_t \langle(\Delta x)^2\rangle_{t=0} \geq \hbar^2 t^2/4m^2}$

e.g., Gaussian wavepacket of earlier [at fixed time ($t = 0$)] : $\langle(\Delta x)^2\rangle_{t=0} = d^2/2$ (say, "small")

(particle localized within $\sim d$ around origin)

– At (much) later time, it spreads (since it is superposition of momenta, (as seen from solving Schroedinger's wave equation), but easier with H - picture : $\langle(\Delta x)^2\rangle_t$ grows with t (uncertainty relation)