Lecture $\boxed{8}$ (Sep. 18, Fri.)

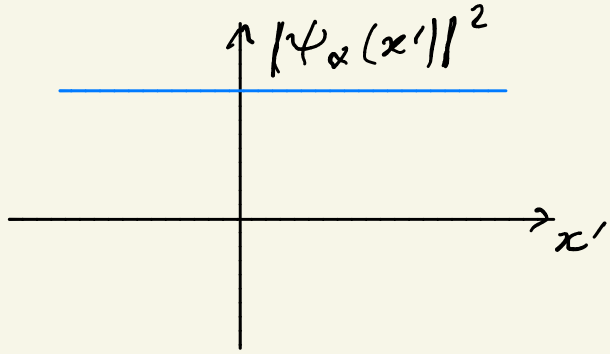
Outline for today

- finish "kinematics" of QM (re-casting into more powerful language / formalism)
- Onto dynamics: $\boxed{\text{Schroedinger equation}}$, but for time evolution or ket (to begin with); then more familiar Schroedinger's wave equation for wavefunction
  (QM analog of classical, i.e., {Newton's / Langrange's / Hamilton's, EOM)

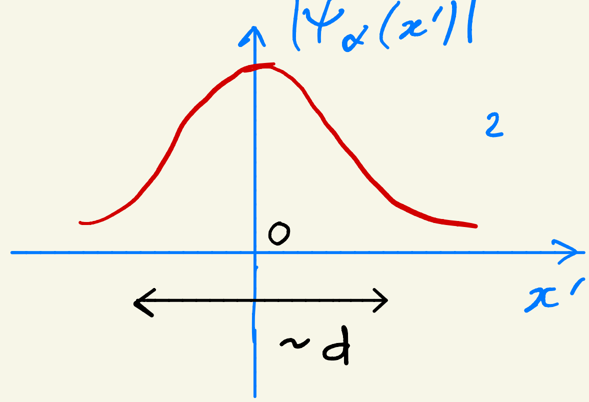
$\boxed{\text{Gaussian}}$ wavepacket

$\boxed{\text{Plane}}$ wave

$\propto \exp(ikx')$



$\boxed{\text{Gaussian}}$



$$\psi_\alpha(x') = \langle x' | \alpha \rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left[ikx' - \frac{x'^2}{2d^2} \right]$$

$|\psi_\alpha(x')|^2$ probability density $\boxed{\text{Gaussian}}$
(localized near origin)

(fixed t: dynamics later)

- $\boxed{\text{expectation}}$ values: $\langle x \rangle = 0$ (by symmetry)

$$\boxed{\langle x^2\rangle} = \int dx'\, x'^2 \,|\langle x'|\alpha\rangle|^2 = \frac{1}{\sqrt{\pi}\, d}\int_{-\infty}^{\infty} dx'\, x'^2 \times \exp\left(\frac{x'^2}{d^2}\right)$$

$$\boxed{= d^2/2}$$

$$\left[\int dx\, e^{-\alpha x^2} = \sqrt{\pi/\alpha} \right.$$

$$\left. \Rightarrow \quad \partial/\partial\alpha \ \ldots \ \int dx\, x^2 e^{-\alpha x^2} = \frac{1}{2}\frac{\sqrt{\pi}}{\alpha\sqrt{\alpha}} \right]$$

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = d^2/2$$

$$\langle p\rangle = \hbar k \ ; \quad \langle p^2\rangle = \hbar^2/(2d) + \hbar^2 k^2$$ (see HW 3.2)

$$\langle(\Delta p)^2\rangle = \hbar^2/(2d^2)$$

$\Rightarrow$ uncertainty product $= \langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \boxed{=} \hbar^2/4$

[(only) Gaussian minimizes it]

F.T. :

$$\boxed{\langle p'|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, \exp\left[-\frac{(p'-\hbar k)^2 d^2}{2\hbar^2}\right]}$$

probability of finding p' is also Gaussian (centered around $\hbar k$) ... with width $\propto 1/d$

$\Rightarrow$ 2 widths inversely related

— $\boxed{d\to\infty}$ limit : plane-wave / delocalized in x' vs. fixed momentum) ... vs. $\boxed{d\to 0}$: δ-function in x' ...

— generalize to 3D : dimension of wave function

$[\psi_\alpha(x')]$ → 1D : $(\text{length})^{-1/2}$

→ 3D : $(\text{length})^{-3/2}$

Outline for $\boxed{\text{dynamics}}$ for today / next week

— time evolution operator (U) related to Hamiltonian

⇒ Schroedinger's equation (SE) for time evolution operator (U) & $|\alpha\rangle$

— $\boxed{\text{Solve}}$ SE for time-evolution operator (U)

— How to actually use above to time evolve state: examples of spin precession; neutrino oscillations (HW 4.1)

— time evolution of expectation values of operators

— time energy uncertainty relation

— Schroedinger ("usual") vs. Heisenberg pictures

$\boxed{\text{Time evolution operator}}$ $[U(t, t_0)]$

$$\underbrace{|\alpha, t_0; t\rangle}_{\text{later time}} \equiv U(t, t_0) \underbrace{|\alpha, t_0; t_0\rangle}_{\text{initial}} = |\alpha, t_0\rangle$$

time is $\boxed{\text{not}}$ an operator (vs. position) in QM

time evolution is like time "displacement" (translation in x')

Properties of $\boxed{U(t, t_0)}$ [like done for $\mathcal{T}(dx')$]

(1). U is unitary (probability conservation:

$\langle \alpha, t_0 | \alpha, t_0 \rangle = 1 \Rightarrow \langle \alpha, t_0, t | \alpha, t_0; t \rangle = 1$)

$\boxed{U^\dagger U = 1}$

(2). composition: $\boxed{U(t_2, t_0) = U(t_2, t_1)\, U(t_1, t_0)}$

(3). $\boxed{\text{infinitesimal}}$ version: $U(t_0 + dt, t_0) = 1 + \ldots$

$$\boxed{U(t_0 + dt) = 1 - dt\, i\Omega}$$ ← Hermitian

(Hamiltonian, H generates "time translation")

... "motivated" by classical mechanics, postulate

$$\boxed{\Omega = H/\hbar}$$

Hamiltonian operator

(same $\boxed{\hbar}$ as in $\boxed{K = p/\hbar}$ so that we recover $\bar{p}/m = \frac{d\bar{x}}{dt}$ as classical limit of QM)

$$\boxed{U(t_0 + dt, t_0) = 1 - iH\,dt/\hbar}$$

$\boxed{\text{Finite}}$ time evolution: divide $\frac{t - t_0}{N} \equiv \Delta t$

... use composition (like for translation) N steps

$$U(t, t_0) = U[\underbrace{t_0 + N\Delta t}_{t},\ t_0 + (N-1)\Delta t] \ldots U(t_0 + \Delta t, t_0)$$

Use $\boxed{\text{each}}$ U on RHS $= (1 - iH\Delta t/\hbar)$

(H is independent of t)

$$\Rightarrow \boxed{U(t, t_0)} = \left(1 - \frac{iH\Delta t}{\hbar}\right)^N \quad \Delta t = \frac{t - t_0}{N}$$

$N \to \infty$, $\Delta t \to 0$

$$= \exp[-iH(t - t_0)/\hbar] = 1 + \left[-\frac{iH(t - t_0)}{\hbar}\right] + \ldots$$

Schroedinger equation for $U(t, t_0)$

$$\boxed{i\hbar \frac{\partial U(t, t_0)}{\partial t} = -\frac{iH}{\hbar}\, i\hbar \left(\exp\left[-\frac{iH(t-t_0)}{\hbar}\right]\right) = H\, U(t, t_0)}$$

SE for $|\alpha\rangle$

independent of t initial

$$i\hbar \frac{\partial U}{\partial t} |\alpha, t_0\rangle = H\, \boxed{U(t, t_0)\, |\alpha, t_0\rangle}$$

$$i\hbar \frac{\partial}{\partial t}\left(U|\alpha, t_0\rangle\right) = H\, |\alpha, t_0; t\rangle$$

only here

$$\Rightarrow \boxed{i\hbar\, \partial/\partial t\, |\alpha, t_0; t\rangle = H\, |\alpha, t_0; t\rangle}$$

In practice, expand $|\alpha\rangle$ in terms of (see e.g.) base kets (eigenkets of Hermitian A) ... $\Rightarrow$

need to know $\boxed{U(t_0, t_0)\, |a'\rangle}$

(I). choose $[H, A] = 0$ ... $\Rightarrow$ $H\, |a'\rangle = \underbrace{E_{a'}}_{\text{eigenvalues of } H} |a'\rangle$

A need not be H

$$\Rightarrow U(t, t_0)\, |a'\rangle = \exp\left[-\frac{iH(t-t_0)}{\hbar}\right] |a'\rangle$$

$$= \exp\left[-\frac{iE_{a'}(t-t_0)}{\hbar}\right] |a'\rangle$$

(II). $|\alpha, t_0 = 0\rangle = \sum_{a'} |a'\rangle \langle a' | \alpha\rangle = \sum_{a'} |a'\rangle\, c_{a'}(t=0)$

$$\text{(III).}\quad |\alpha, t_0; t\rangle = U(t, t_0)|\alpha, t_0\rangle$$

$$= U(t, t_0)\left(\sum_{a'} |a'\rangle c_a(t=0)\right)$$

$$= \sum |a'\rangle c_a(t=0) \exp\left[-\frac{iE_{a'}t}{\hbar}\right]$$

$$\boxed{c_{a'}(t) = c_{a'}(t=0)\exp\left(-\frac{iE_{a'}t}{\hbar}\right)}$$

$\Rightarrow |c_{a'}(t)|^2 = |c_{a'}(t=0)|^2$: probability to measure a' constant in time (see e.g.) ... but relative phases between components change, since $E_{a'}$ are different ... $\Rightarrow$ state does evolve non-trivially (in general)

– special / warm-up case: $|\alpha, t_0\rangle = |a'\rangle$, then

$$|a', t_0 = 0, t\rangle = |a'\rangle \exp(-iE_{a'}t/\hbar)$$

$\Rightarrow$ observable which is compatible with H is constant of motion

Expectation values

– warm-up: initial ket is eigenket of A ($[A, H] = 0$) ~~&~~ not commuting with H

$$\langle B \rangle = \underbrace{\langle a' | U^\dagger(t, t_0)}_{\text{new bra}} \cdot B \cdot \underbrace{U(t, t_0)|a'\rangle}_{\text{new ket}}$$

use $U(t, t_0)|a'\rangle = \exp[-iE_a(t-t_0)(\tfrac{1}{\hbar})]|a'\rangle$

$$\ldots = \langle a' | B | a' \rangle$$ (stationary state expectation value is constant)

Onto Superposition as initial ket: $\langle B \rangle = \left[\sum_{a'} c_{a'}^* \langle a' | \exp\left(\frac{i E_{a'} t}{\hbar}\right) \right]$ new bra

$$\cdot B \cdot \left[\sum_{a''} c_{a''} \exp\left(\frac{-i E_{a''} t}{\hbar}\right) \right]$$ new ket

$$= \sum_{a'} \sum_{a''} c_{a'}^* c_{a''} \langle a' | B | a'' \rangle \exp\left[\frac{i(E_{a''} - E_{a'}) t}{\hbar}\right] \boxed{\neq} \text{ constant } (E_{a'} \neq E_{a''})$$

sum of oscillating terms

Exception (see e.g.): $[B, A] = 0 \Rightarrow \langle a' | B | a'' \rangle = \delta_{a'a''} b'$ ... no oscillating term; $\langle B \rangle$ at time $t = \sum_{b'} |c_{b'}|^2 b'$ = constant

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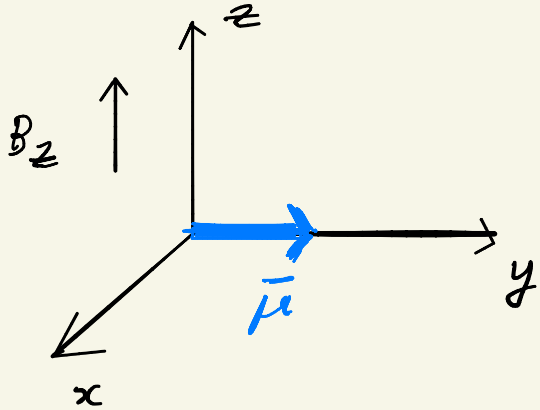
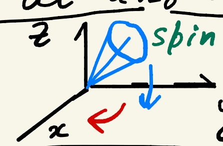
$\boxed{\text{spin precession}}$ $\boxed{\text{HW 4.2, 4.3}}$



torque $\bar{\tau} = \bar{\mu} \times \bar{B}$ along x initially $\Rightarrow$ tries to align $\bar{\mu}$ with $\bar{B}$ ... but $\bar{\mu}$ comes with spin $\boxed{\text{angular momentum}}$

$\boxed{\text{Classically}}$ e.g., spinning top let go at angle to vertical (gravity) direction: precesses as it falls



$H = -\bar{\mu} \cdot \bar{B}$; $|\bar{\mu}| = \frac{e\hbar}{2 m_e c} = |\bar{S}| \frac{e}{m_e c}$; $\bar{B} = B\hat{z}$

(neglect kinetic energy)

$$\boxed{H = -\frac{eB}{m_e c} S_z} \Rightarrow [H, \overset{A \text{ of general formalism}}{S_z}] = 0 \quad (e < 0)$$

(I).

$E_\pm = \mp e\hbar B / 2 m_e c$ for $|S_z; \pm\rangle$

$\boxed{\omega = |e| B / (m_e c)}$; $(E_+ - E_-) = \hbar\omega$; $\boxed{H = \omega S_z}$

$$|\alpha, t_0\rangle = c_+ |+\rangle + c_- |-\rangle \longrightarrow |\alpha, t_0 = 0; t\rangle$$

(II). $\begin{pmatrix}\text{use general} \\ \text{formula}\end{pmatrix}$ $= c_+ \exp\left(\frac{-i\omega t}{2}\right)|+\rangle$ (III).

$$+ c_- \exp\left(\frac{+i\omega t}{2}\right)|-\rangle$$

e.g., initially $|S_y;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$

probability to be found in $|S_y;\pm\rangle$ at time t

$$= \left|\underbrace{\left(\frac{1}{\sqrt{2}}\left\{\langle +| \mp i \langle -|\right\}\right)}_{\langle S_y;\pm|} \cdot \frac{1}{\sqrt{2}}\left(\left[|+\rangle \exp\left(\frac{-i\omega t}{2}\right) + i|-\rangle \exp\left(\frac{+i\omega t}{2}\right)\right]\right)\right|^2$$

new ket

$$= \left|\frac{1}{2}\exp\left(\frac{-i\omega t}{2}\right) \pm \frac{1}{2}\exp\left(\frac{+i\omega t}{2}\right)\right|^2 = \begin{cases}\cos^2\frac{\omega t}{2} & \text{for } |S_y;+\rangle \\ \sin^2\frac{\omega t}{2} & \text{for } |S_y;-\rangle\end{cases}$$

$\Rightarrow$ non-zero probability for initial $|S_y;+\rangle$ to be found as $|S_y;-\rangle$ later (related probability to be found as $|S_y;+\rangle$ reduced)

$$\langle S_y\rangle = \underbrace{+\hbar/2}_{\text{eigenvalue}}\ \underbrace{\cos^2\frac{\omega t}{2}}_{\text{probability}} - \frac{\hbar}{2}\sin^2\frac{\omega t}{2} = \frac{\hbar}{2}\cos\omega t$$

... $\langle S_x\rangle = \hbar/2 \sin\omega t$ & $\langle S_z\rangle = 0$ (constant: as per general result) $\Rightarrow$

$\boxed{\text{spin } \boxed{\text{precesses}} \text{ in } x\text{-}y \text{ plane}}$ ("like" classical spinning top)

(probability to measure S_z to be $\pm\hbar/2$ $= 1/2$ at all times, as per general result)