Lecture $\boxed{7}$ (Sept. 16, Wed.)

- Review part of course: framing known ideas in more general / mathematical language
- "kinematics" of QM first: state - ket (vector); observable - operator (position / spin) ...
- dynamics (Schroedinger equation) next

Outline for today

- successive translations in x & y ... $\Rightarrow [p_x, p_y]$
- summary of commutation relations ("preview" of rotations, angular momentum operator)
- **Note**: so far, independent of representation of x, p operators / state-ket (no wavefunction or $p = i\hbar \partial/\partial x'$ yet: that's next!)
- Onto more "playing with" position / momentum ("connecting to" UG-level QM)
  - wavefunction of state in position & momentum spaces $[\psi_\alpha(x')$ and $\phi_\alpha(p')]$
  - momentum operator in position basis $\left[p \doteq i\hbar\, \partial/\partial x' \right]$
  - Gaussian wavepackets

$\boxed{\text{Successive}}$ translations in $\boxed{x \text{ and } y}$ ...

commute (rotations do not)

$$T(\Delta y' \hat{y})\, T(\Delta x' \hat{x}) = T(\Delta x', \hat{x})\, T(\Delta y', \hat{y})$$

$$\ldots \Rightarrow [p_x, p_y] = 0$$

(translation group in 3D is Abelian)

$\boxed{\text{Summary}}$ $[x_i, x_j] = 0 = [p_i, p_j]$; $[x_i, p_j]$
$= i\hbar \delta_{ij}$

... obtained by (1) properties of translations and (2) momentum is generator of translation

... generalize to rotations (1) rotations don't commute and (2) angular momentum generates rotations ... including spin (and orbital)

$\hookrightarrow$ no classical analog

(cf. position, momentum : linear and orbital angular)

vs. Dirac's "rule" : classical Poisson brackets

$\longrightarrow$ $\dfrac{\boxed{\text{commutator}}\text{ of (quantum) operators}}{i\hbar}$

works for linear & orbital angular momentum but not for spin

Wavefunction in position space

- probability for particle described by $|\alpha\rangle$ to be found in interval dx' around x'

$$= |\langle x'|\alpha\rangle|^2 dx' \quad \text{(based on general postulate)}$$

$$\Rightarrow \boxed{\psi_\alpha(x')} \text{ [wavefunction for } |\alpha\rangle \text{ in position representation]} \equiv \langle x'|\alpha\rangle$$

[again, one example (basis) of general concept: basis expansion coefficient, $c_{a'} \equiv \langle a'|\alpha\rangle$ gives probability to measure a' (here, use $|x'\rangle$ as $|a'\rangle$)]

- $\langle\beta|\alpha\rangle$ = probability amplitude for $|\alpha\rangle$ to be "found" in $|\beta\rangle$

$$= \int dx' \langle\beta|x'\rangle\langle x'|\alpha\rangle = \int dx' \, \psi_\beta^*(x')\psi_\alpha(x')$$

$\therefore$ as expected

- $$\underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')} = \sum_{a'} \langle x'|a'\rangle\langle a'|\alpha\rangle = \sum_{a'} c_{a'} \underbrace{u_{a'}(x')}_{\langle x'|a'\rangle}$$

(position-space) wavefunction of $|a'\rangle$

- $$\langle\beta|A|\alpha\rangle \overset{\text{can show}}{=} \int dx' dx'' \, \psi_\beta^*(x') \langle x'|A|x''\rangle \psi_\alpha(x'')$$

$$A = f(\cancel{x}) \text{ operator} = \sum_n c_n x^n$$

$$\langle x'|x^n|x''\rangle = \langle x'|\underset{\text{eigenvalue}}{x''^n}|x''\rangle = x''^n \delta(x' - x'')$$

Combining above 3 equations: $\langle \beta | A = f(x) | \alpha \rangle = \int dx' \, \psi_\beta^*(x') f(x') \psi_\alpha(x')$

operator

number

Momentum operator in position basis

$$p \doteq -i \, \partial/\partial x' \, \hbar$$

$$\left(1 - i \, p/\hbar \, \Delta x'\right) |\alpha\rangle = \int dx' \, T(\Delta x') |x'\rangle\langle x'|\alpha\rangle$$

$|\alpha\rangle$

$$= \int dx' \, |x' + \Delta x'\rangle\langle x'|\alpha\rangle$$

Taylor expand (small $\Delta x'$)

$$\overset{x' \to x' - \Delta x'}{=} \int dx' \, |x'\rangle\langle x' - \Delta x'|\alpha\rangle$$

drop $(\Delta x')^2$ on both sides

$$= \underbrace{\int dx' |x'\rangle\langle x'|\alpha\rangle}_{|\alpha\rangle} - \int dx \, |x'\rangle (\Delta x') \frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

$$\Rightarrow \langle x' | p | \alpha \rangle = \langle x' | \int dx \, |x'\rangle \left[-i\hbar \, \partial/\partial x' \langle x'|\alpha\rangle\right]$$

$$= -i\hbar \, \partial/\partial x' \underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')}$$

... as expected

(plug later)

$$\langle x' | p | x'' \rangle = -i\hbar \, \partial/\partial x' \, \delta(x' - x'')$$

$$\langle \beta | p | \alpha \rangle = \int dx' \, \psi_\beta^*(x') \left[-i\hbar \, \partial/\partial x' \, \psi_\alpha(x')\right]$$

... all derived from properties of p operator in general

Wavefunction in momentum space

$p|p'\rangle = p'|p'\rangle$ ... probability for measurement to give $p' \pm dp'/2$ is $|\langle p'|\alpha\rangle|^2 dp'$

$\boxed{\langle p' | \alpha \rangle = \phi_{\alpha}(p')}$ ... related to $\psi_\alpha(x')$ by Fourier transform

show by use of change of basis (in general): need $\langle x' | p' \rangle$ "old" new

position-space wavefunction of $|p'\rangle$ → momentum eigenket

⇒ expect ... plane-wave: derive it

$|\alpha\rangle$ earlier

$$\langle x' | p | p' \rangle = -i\hbar \, \partial/\partial x' \, \langle x' | p' \rangle$$

$$= p' \langle x | p' \rangle$$

... solve (DE) by

$$\langle x' | p' \rangle = N \exp\left(\frac{i p' x'}{\hbar}\right)$$

normalization

$$= \frac{1}{\sqrt{2\pi\hbar}} \exp(i p' x'/\hbar)$$

$$\underbrace{\langle x' | \alpha \rangle}_{\psi_\alpha(x')} = \int dp' \underbrace{\langle x' | p' \rangle}_{\frac{1}{\sqrt{2\pi\hbar}} \exp(ip'x'/\hbar)} \overbrace{\underbrace{\langle p' | \alpha \rangle}_{\phi_\alpha(p')}}^{1}$$

$$\psi_\alpha(x') = \frac{1}{\sqrt{2\pi\hbar}} \int dp' \, \phi_\alpha(p') \exp\left(i\frac{p'x'}{\hbar}\right)$$

$\langle p' | x' \rangle = \langle x' | p' \rangle^*$

$$\phi_\alpha(p') = 1/\sqrt{2\pi\hbar} \int dx' \exp(-ip'x'/\hbar) \, \psi_\alpha(x')$$

Normalization from $\langle x' | x'' \rangle = \boxed{\delta(x' - x'')}$

$$= \int dp' \, \langle x' | p' \rangle \langle p' | x'' \rangle = |N|^2 \times \int dp' \exp\left(\frac{i p'(x' - x'')}{\hbar}\right)$$

$$= |N|^2 \, 2\pi\hbar \, \boxed{\delta(x' - x'')}$$

So, $\boxed{N = 1/\sqrt{2\pi\hbar}}$

Position operator in momentum basis : HW 3.4