Lecture $\boxed{6}$ (Sept. 14, Mon.)

General $\boxed{\text{theme}}$: re-cast concepts of position / momentum operators, wavefunction in more systematic / mathematical language

Outline for today

- translation (infinitesimal) operator [classical mechanics]
→ momentum as generator of translation
- canonical commutation relation: $\boxed{[x, p]}$
- successive translations along x and y: $\boxed{[p_x, p_y]}$

x

Infinitesimal $\boxed{\text{translation}}$ operator

state localized at $\underbrace{x'}_{\text{old}} \longrightarrow \underbrace{(x' + dx')}_{\text{new}}$

$$T(dx')\,|x'\rangle = |x' + dx'\rangle$$

$\boxed{|x'\rangle \text{ not eigenket of } T(dx')}$

$$\boxed{T(dx')\,|\alpha\rangle} = T(dx') \int dx''\, \underbrace{\langle x''|\alpha\rangle}_{\psi_{old}(x'')}\, \overbrace{|x''\rangle}^{\text{eigen-ket}}$$

acts

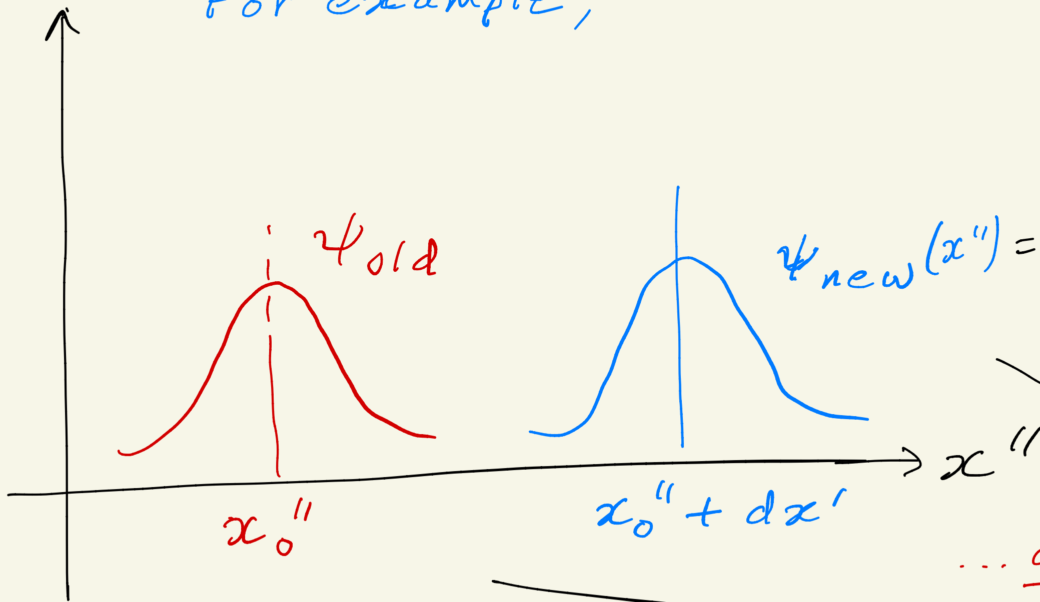
$$= \int dx''\, \langle x''|\alpha\rangle\, |x'' + dx'\rangle \qquad x'' \to x'' - dx'$$

dummy

$$= \int dx''\, \underbrace{\langle x'' - dx'|\alpha\rangle}_{\psi_{new}(x'')}\, |x''\rangle$$

$$\psi_{new}(x'') = \psi_{old}(x'' - dx')$$

For example,



$\psi_{new}(x'') = \underbrace{\psi_{old}(x'' - dx')}_{\text{peaks at } x_0' + dx'}$

... as expected ...

$\boxed{\text{Properties}}$ of $T(dx')$

(1) $\langle \alpha | \alpha \rangle = 1 = \langle \alpha | T^\dagger(dx') T(dx') | \alpha \rangle$

$\Rightarrow \boxed{T^\dagger(dx')\, T(dx') = 1}$ (unitary)

(2) $T(dx')\, T(dx'') = T(dx' + dx'')$

(3) $T(-dx') = T^{-1}(dx')$

(4) $\lim_{dx' \to 0} T(dx') = \mathbb{1}$

$\Rightarrow \boxed{T(dx') = \mathbb{1} - i K\, dx'}$ ... (a)

$\boxed{K \text{ is Hermitian}}$ (proof in Sakurai / "informal" HW)

[$T(dx')$ not Hermitian]

K operator generates infinitesimal translation

... reminds us of $\boxed{\text{classical}}$ mechanics:

$\boxed{p}$ generates infinitesimal translation (classical canonical variables, not operators): Phys 601 or Goldstein: p. 403, Eq. 9.108

Postulate / definition :

$$\boxed{T(dx') = 1 - i\frac{p\,dx'}{\hbar}} \quad \cdots (6)$$

$$\boxed{K = \frac{p}{\hbar}} \text{ (operator)}$$

to match dimensions

("motivated" by de Broglie relation)

— x —

Finite translation :

vs. dx' earlier

$$T(\Delta x')\,|x'\rangle = |x' + \Delta x'\rangle$$

$$T(\Delta x') = \lim_{N\to\infty} \Big(1 - i p \underbrace{\frac{\Delta x'}{N}}_{dx'} \frac{1}{\hbar}\Big)^N$$

$$= \exp\left(-\frac{i p \Delta x}{\hbar}\right)$$

e.g. HW 2.4

in general : $\exp(X) \equiv 1 + X + \frac{X^2}{2} + \cdots$

— x —

Use $[p, T(\Delta x')] = 0$; $p\,|p'\rangle = \underbrace{p'}_{\text{eigenvalue}}\,\overbrace{|p'\rangle}^{\text{eigenket}}$

$$T(dx')\underbrace{|p'\rangle}_{} = \left(1 - i\frac{p'\,dx'}{\hbar}\right)|p'\rangle$$

eigenket of $\boxed{T(\Delta x')}$

with eigenvalue complex [$T(dx')$ is not Hermitian]

— x —

Commutation relation

claim $[x, T(dx')] = dx'$

$\left(\frac{1^{st}}{\text{order}}\right)$ $x\,T(dx')\,|x'\rangle = x\,|x'+dx'\rangle = (x'+dx')\,|x'+dx'\rangle$

$\left(\frac{\text{other}}{\text{way}}\right)$ $T(dx')\,x\,|x'\rangle = x'\,T(dx')\,|x'\rangle = x'\,|x'+dx'\rangle$

subtract $\Rightarrow$

$|x' + dx'\rangle = |x'\rangle + (dx')(\text{ket})$

$$[x, T(dx')]\,|x'\rangle = dx'\,|x'+dx'\rangle$$

$$\approx dx'\,|x'\rangle \quad \text{drop } (dx')^2$$

$\rightarrow$ complete set

$$\Rightarrow [x, T(dx')] = dx'$$

Plug $T(dx')$ in terms of p $\Rightarrow$ $\boxed{[x, p] = i\hbar}$

$\Rightarrow$ uncertainty relation: $\boxed{\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \hbar^2/4}$

Generalize to 3d : $\boxed{[x_i, p_j] = i\hbar\,\delta_{ij}}$

Next, $[p_i, p_j]$ ...

[Also, $[x, G(p)]$ in problem 1.26 of Sakurai : used later]