Lecture $\boxed{5}$ (Sept. 11, Friday)

Last time: compatible observables / commuting operators ($[A, B] = 0$)

Complete set of simultaneous eigenkets:

$A|a', b'\rangle = a'|a', b'\rangle \; ; \; B|a', b'\rangle = b'|a', b'\rangle$

$\boxed{\text{Non}}$-degenerate eigenvalues: "b'" label in $|a', b'\rangle$

$\boxed{\text{not}}$ needed, since given a', can get b' $(= \langle a'|B|a'\rangle)$

For degenerate case, we do need b' label, e.g., for orbital angular momentum, $[L^2, L_z] = 0$, with L^2 eigenvalues being $\ell(\ell+1)\hbar^2$ (ℓ is integer) and L_z eigenvalues are $m_\ell \hbar$, $m_\ell = -\ell, -\ell+1 \ldots 0 \ldots \ell-1, \ell$ for each ℓ $\Rightarrow$ state characterized by ℓ and m_ℓ

(just ℓ is not enough)

Outline for today

— Incompatible observables / non-commuting operators: $[A, B] \neq 0 \Rightarrow |a'\rangle$ "different" than $|b'\rangle$

$\longrightarrow$ uncertainty relations (e.g. x, p_x)

$\longrightarrow$ change of basis (from $|a'\rangle$ to $|b'\rangle$)

— $\boxed{\text{Continuous}}$ spectra of eigenvalues: position and momentum (vs. discrete thus far, e.g. spin-$\frac{1}{2}$ system)

$\boxed{[A, B] \neq 0}$: do not have a complete set of simultaneous eigenkets

$\boxed{\text{Proof}}$ by contradiction : $A|a', b'\rangle = a'|a', b'\rangle$

& $B|a', b'\rangle = b'|a', b'\rangle$ $\quad$ simultaneous eigenket

$AB|a', b'\rangle = A\, b'|a', b'\rangle = a'b'|a', b'\rangle$

... $BA|a', b'\rangle = b'a'|a', b'\rangle$

$(AB - BA)\underbrace{|a', b'\rangle}_{\text{assume complete set}} = 0 \Rightarrow AB - BA = 0$

$[A, B] = 0$ ... contradiction

(not complete)

– $\boxed{\text{subspace}}$ of all eigenkets can be simultaneous eigenkets, e.g., $[L_x, L_z] \neq 0$, but $\boxed{l = 0}$ (s-wave) is eigenstate of L_x & L_z

weirdness for $\boxed{\text{successive}}$ measurements (e.g. with SG apparatus for spin $1/2$, see HW $\boxed{2.2}$)

probability to get $\boxed{c'}$ starting with $|a'\rangle$ (normalized) with $\boxed{B}$ measured to be $\boxed{b'}$ (fixed)

$$= |\langle b'|a'\rangle|^2 \; |\langle c'|b'\rangle|^2$$

... Next, sum over b' (still measuring it):

probability to get $c' = \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle \langle a'|b'\rangle\langle b'|c'\rangle$

... compare to case with $\boxed{\text{no}}$ B measurement at all $= |\langle c'|a'\rangle|^2$

$$= \sum_{b'}\sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle$$

... 2 probabilities different, unless $[A, B] = 0$ or $[B, C]$ or $[A, C] = 0$

$\Rightarrow$ outcome of C measurement depends on whether (or not) we measure B (even if we sum over all b')

Uncertainty relation (generalized)

more in HW $\boxed{3.2}$

$$\Delta A \equiv A - \underbrace{\langle A \rangle}_{\langle \alpha | A | \alpha \rangle}$$

$\langle A \rangle$ is real : $\langle a' | X | a'' \rangle = \langle a'' | X^\dagger | a' \rangle^*$ in general

Here $A = A^\dagger$ (Hermitian)

& $|a'\rangle = |a''\rangle$

$$\underbrace{\langle (\Delta A)^2 \rangle}_{\text{dispersion of } A} = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

... vanishes when $|\alpha\rangle = |a'\rangle$ (easy, "informal HW")

$\rightarrow$ dispersion measuring "uncertainty" in A, e.g.,

for $|S_z; +\rangle$ state, $\langle (\Delta S_x)^2 \rangle = \hbar^2/4$ (fuzzy)

while $\langle (\Delta S_z)^2 \rangle = 0$ (sharp)

$$\Rightarrow \boxed{\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2}$$

(proof in Sakurai)

Change of basis

Idea : same ket space spanned by $|a'\rangle$

& $|b'\rangle$ (different)

(Again, $A|a'\rangle = a'|a'\rangle$; $B|b'\rangle = b'|b'\rangle$

$|a^{(k)}\rangle : k = 1, 2, \dots N$ $\qquad |b^{(l)}\rangle ; l = 1, 2 \dots$

$$|\alpha\rangle = \sum_k |a^{(k)}\rangle \langle a^{(k)}|\alpha\rangle \quad \cdots (1)$$

given

$$= \sum_\ell |b^{(\ell)}\rangle \langle b^{(\ell)}|\alpha\rangle \quad \cdots (2)$$

want (e.g. measuring B)

$$= \sum_{b'} \sum_{a'} |a'\rangle\langle a'|b'\rangle \langle b'|\alpha\rangle$$

want

compare

$$\Rightarrow \quad \sum_\ell \langle a^{(k)}|b^{(\ell)}\rangle \langle b^{(\ell)}|\alpha\rangle = \langle a^{(k)}|\alpha\rangle \quad \cdots (3)$$

want   given

need these (B eigenket expanded again: in $|a'\rangle$ basis)

$\Rightarrow$ expand $|b'\rangle$ in $|a'\rangle$ basis :

$$|b^{(\ell)}\rangle = \sum_k |a^{(k)}\rangle \langle a^{(k)}|b^{(\ell)}\rangle \quad \cdots (4)$$

need

– How to get $\langle a^{(k)}|b^{(\ell)}\rangle$

go to

– matrix representation : $\underbrace{\langle a''|}_{\text{row}} B \overbrace{|a'\rangle}^{\text{column}}$ (not diagonal)

diagonalization:

$\longrightarrow$ find eigenvectors/values : $\ell = 1, 2 \ldots N$ (e.g. in HW 3.1)

$$(B \text{ matrix}) \begin{pmatrix} c_1^{(\ell)} \\ c_2^{(\ell)} \\ \vdots \\ c_N^{(\ell)} \end{pmatrix} = \lambda^{(\ell)} \begin{pmatrix} c_1^{(\ell)} \\ \vdots \\ c_N^{(\ell)} \end{pmatrix}$$

$C^{(\ell)}_{1 \ldots N}$ are $\langle a^{(1 \ldots N)}|b^{(\ell)}\rangle$ (needed)

– need to invert (3) : define U (operator)

by matrix elements : $\boxed{\langle a^{(k)}|U|a^{(\ell)}\rangle \equiv \langle a^{(k)}|b^{(\ell)}\rangle} \quad \cdots (5)$

($\Rightarrow$ ℓ^{th} column of U is $|b^{(\ell)}\rangle$ in $|a'\rangle$ basis)

Eq (3) is (U matrix) $\begin{pmatrix} \text{new column} \\ \text{of } |\alpha\rangle \end{pmatrix}$ = old $|\alpha\rangle$ ...(3')

in $|b'\rangle$ basis $|a'\rangle$ basis

Proof of U is unitary:

$$\langle a^{(k)} | U^\dagger | a^{(\ell)} \rangle$$ general

use form of U

$$= \langle a^{(\ell)} | U | a^{(k)} \rangle^* = \langle a^{(\ell)} | b^{(k)} \rangle^*$$
$$= \langle b^{(k)} | a^{(\ell)} \rangle$$

$$\Rightarrow \langle a^{(k)} | U^\dagger U | a^{(\ell)} \rangle$$
$$= \sum_m \langle a^{(k)} | U^\dagger | a^{(m)} \rangle \langle a^{(m)} | U | a^{(\ell)} \rangle$$
$$= \sum_m \langle b^{(k)} | a^{(m)} \rangle \langle a^{(m)} | b^{(\ell)} \rangle$$
$$= \langle b^{(k)} | b^{(\ell)} \rangle = \delta_{k\ell}$$

$\Rightarrow$ (3) or (3') gives (upon $U^\dagger$ from left)

$$\boxed{\text{new } |\alpha\rangle \text{ column} = U^\dagger \text{ old } |\alpha\rangle}$$

U given "matrix-independently":

$$\langle a^{(k)} | U | a^{(\ell)} \rangle = \langle a^{(k)} | b^{(m)} \rangle \delta_{m\ell}$$
$$= \sum_m \langle a_m | a_\ell \rangle \langle a^{(k)} | b^{(m)} \rangle$$
$$= \langle a^{(k)} | \sum_m | b^{(m)} \rangle \langle a^{(m)} | a^{(\ell)} \rangle$$

$$\boxed{U = \sum_m | b^{(m)} \rangle \langle a^{(m)} |}$$

$$\boxed{| b^{(\ell)} \rangle = U | a^{(\ell)} \rangle}$$ (easy "informal HW")

General operator $\{X\}$ in $|a'\rangle$ vs. $|b'\rangle$ basis

$$\langle b^{(k)} | X | b^{(l)} \rangle = \sum_m \sum_n \underbrace{\langle b^{(k)} | a^{(m)} \rangle}_{U} \overbrace{\langle a^{(m)} | X | a^{(n)} \rangle}^{X \text{ in old basis}} \cdot \underbrace{\langle a^{(n)} | b^{(l)} \rangle}_{U}$$

(new)

$$\Rightarrow \boxed{X \text{ in new} = U^{\dagger} X_{old} U}$$

(similarity transformation)

... but $\boxed{\text{trace}}$ unchanged (see Sakurai or "informal HW")

tr X (in general) = Σ diagonal elements in matrix representation

$$= \sum_{a'} \langle a' | X | a' \rangle$$

$\boxed{\text{Unitary equivalent}}$ observables

$\tilde{A} \equiv U A U^{-1}$ is unitary transform of A;

A & UAU^{-1} are unitary equivalent

$$-\ \underbrace{(U A U^{-1})}_{\tilde{A}} \underbrace{}_{1} (U | a^{(l)} \rangle) = a^{(l)} (U | a^{(l)} \rangle)$$

A eigenvalues (of $\tilde{A}$ here)

eigenket $|b'\rangle$ (of $\tilde{A}$ here)

$\boxed{\text{Onto}}$

$\int$ Continuous spectra : generalize discrete...

$$\langle a' | a'' \rangle = \delta_{a'a''} \longrightarrow \langle \xi' | \xi'' \rangle = \delta(\xi' - \xi'')$$

$$(\xi | \xi' \rangle = \xi' | \xi' \rangle)$$

$$\sum_{a'} |a'\rangle\langle a'| = \mathbb{1} \longrightarrow \int d\xi' \, |\xi'\rangle\langle\xi'| = \mathbb{1} \; \ldots$$

Position $\boxed{\text{eigenkets}}$

$$x\,|x'\rangle = x'\,|x'\rangle$$

x: operator; x': eigenvalue | $|x'\rangle$: eigenket

$$\boxed{|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'|\alpha\rangle}$$

e.g., device turns on when particle at $\boxed{x'}$, collapses $|\alpha\rangle$ into $\boxed{|x'\rangle}$, like S_z …

… practically, particle is $\left(x' - \frac{\Delta}{2}, x' + \frac{\Delta}{2}\right)$ so (Δ small)

(cf. discrete case)

$$|\alpha\rangle = \int dx'' |x''\rangle\langle x''|\alpha\rangle \xrightarrow{\text{measurement}} \int_{x'-\Delta/2}^{x'+\Delta/2} dx'' \, |x''\rangle\langle x''|\alpha\rangle$$

- assume Δ small enough so $\langle x''|\alpha\rangle$ constant within this region: probability of detection is $|\langle x'|\alpha\rangle|^2 \underbrace{dx'}_{\Delta}$ (like $|\langle a'|\alpha\rangle|^2$ of discrete)

- total probability $= 1 = \int_{-\infty}^{\infty} dx' |\langle x'|\alpha\rangle|^2 = \int_{-\infty}^{\infty} dx' \langle\alpha|x'\rangle \underbrace{\langle x'|\alpha\rangle}$

x-representation of wavefunction of $|\alpha\rangle$

- extend to 3d: assume $\boxed{[x_i, x_j] = 0}$

$x_{1,2,3} = x, y, z$