Lecture $\boxed{43}$, Dec. 14 (Mon.)

Outline for last lecture

- Based on time-reversal properties of angular momentum eigenkets, derive those of expectation values for these states of time-reversal even/odd operators
- more physical consequences of time-reversal transformation
- e.g. of using symmetries for getting selection rule

- For $\overset{\times}{A}$ being even/odd under time reversal (in general)

$$\langle \alpha | A | \alpha \rangle = \pm \langle \tilde{\alpha} | A \underbrace{| \tilde{\alpha} \rangle}_{\Theta | \alpha \rangle}$$

- Plug $|\alpha\rangle = |\alpha, \underbrace{j, m}_{\text{angular momentum eigenket}}\rangle$

α: radial quantum number

with $\Theta \, | j, m \rangle = (i)^{2m} \, | j, -m \rangle$

$\leftrightarrow \langle j, -m | \left[(i)^{2m} \right]^*$

to get

$$\boxed{\langle \alpha, j, m | A | \alpha, j, m \rangle = \pm \langle \alpha, j, -m | A | \alpha, j, -m \rangle}$$

- More specifically, A is spherical tensor : $T_q^{(k)}$: suffices to get matrix elements for $q=0$, since others related to it "geometrically" using Wigner-Eckart theorem :

- $T^{(k)}$ (Hermitian) even/odd under time-reversal based on $q=0$:

$$\Theta T_{q=0}^{(k)} \Theta^{-1} = \pm T_{q=0}^{(k)} \Rightarrow$$

$$\langle \alpha, j, m | T_{q=0}^{(k)} | \alpha, j, m \rangle = \pm \langle \alpha, j, -m | T_0^{(k)} | \alpha, j, -m \rangle$$

- Now, $| j, -m \rangle$ "related to" $| j, +m \rangle$ by rotation operator and we know how $T_0^{(k)}$ transforms under rotations $\Rightarrow$ (details in Sakurai)

$$\langle \alpha, j, m | T_0^{(k)} | \alpha, j, m\rangle = \pm(-1)^k \langle \alpha, j, m | T_0^{(k)} | \alpha, j, m\rangle$$

not $-m$!

i.e., $\langle \alpha, j, m | T_0^{(k)} | \alpha, j, m\rangle \left[1 \mp (-1)^k\right] = 0$

$T_0^{(k)}$ is even/odd...

$\Rightarrow$ $\boxed{\langle \alpha, j, m | T_0^{(k)} | \alpha, j, m\rangle \text{ vanishes unless } \pm(-1)^k = 1 \hookrightarrow T_{q=0}^{(k)} \text{ even/odd}}$

e.g., $T_0^{(k)} = \bar{x}$ $(k = 1)$, with $\bar{x}$ even

$\Rightarrow \langle j, m | \bar{x} | j, m\rangle = 0$: EDM vanishes for angular momentum eigenket, cf. earlier, vanishes if parity eigenket

— complementary: $|\alpha, j, m\rangle$ need not have definite parity, e.g., spin-$1/2$ particle with $l = 0$ (s-wave) or $l = 1$ (p-wave) can have $j = 1/2$:

$$\underbrace{|j=1/2, m\rangle}_{\text{angular momentum eigenket}} = c_s|s_{1/2}\rangle + c_p|p_{1/2}\rangle$$

s: $l=0$, $1/2$: j; p: $l=1$, $1/2$: j

i.e., not parity eigenket (s or p are even or odd)

Physical implication of time-reversal symmetry: even if

$[\Theta, H]=0$, $\Theta\, U(t,t_0) \neq U(t,t_0)$,

since $U \simeq 1 - iH(t-t_0)/\hbar \cdots$

(cf. other symmetries, including parity) ...

... but do get wavefunction real (no degeneracy), without spin

— More mileage including spin:

$[\Theta, H] = 0 \Rightarrow |n\rangle$ & $\Theta|n\rangle$ have energy eigenvalue

- If they are same state (no degeneracy), then $\Theta|n\rangle = e^{i\delta}|n\rangle \Rightarrow$

$$\Theta^2|n\rangle = \Theta\left(e^{i\delta}|n\rangle\right) = e^{-i\delta}\Theta|n\rangle$$
$$= e^{-i\delta}e^{i\delta}|n\rangle$$

i.e., $\Theta^2|n\rangle = +|n\rangle$ ... which is not valid for half-integer j ($\Theta^2 = -\mathbb{1}$ for it)

$\Rightarrow |n\rangle$ and $\Theta|n\rangle$ are different: there is Kramer's degeneracy,

e.g., $V = e\phi(x)$ (external electrostatic field)

$\Rightarrow [\Theta, H] = 0$ since $[\Theta, x] = 0$
$= [\Theta, \text{function of } x]$

$\Rightarrow$ with odd number of electrons (total j - including orbital & spin - is half-integer) in any $\bar{E}$, each energy level at least 2-fold degenerate

- However, with external magnetic field (assume not changed by time-reversal), $H \ni \bar{S} \cdot \bar{B}$; $\bar{p} \cdot \bar{A} + \bar{A} \cdot \bar{p}$ $(\bar{B} = \bar{\nabla} \times \bar{A})$, where $\bar{S}, \bar{p}$ odd under time-reversal, so $\Theta H \neq H\Theta \Rightarrow$ Kramer's degeneracy for odd number of electrons removed by external $\bar{B}$

e.g. spin-½ system: $\Theta |+\rangle \sim |-\rangle$, but energies in $\bar{B}$ different for $|\pm\rangle$

Example of angular momentum conservation combined with Bose-Einstein statistics (identical integer-spin particles)

spin-1 particle (vector) $\xrightarrow{\text{decay}}$ 2 identical spin-0 (scalar) ?

(creation/annihilation of particles needs relativistic QM / Quantum field theory, but here simply assume some interaction Hamiltonian — rotationally invariant — can do it : angular momentum conserved between initial & final state)

— Final state : 2 $\boxed{\text{identical}}$ bosons (integer spin particles) ⇒ $\boxed{\text{overall}}$ wavefunction unchanged/symmetric under exchange of all quantum numbers of 2 particles

— $\boxed{2}$ "sub"-spaces of quantum state (in general) : spin & orbital angular momentum

- Here no spin for final state particles, hence orbital part of wavefunction ψ must be symmetric

- Now, ψ is function of $\boxed{\bar{r} = \bar{r}_1 - \bar{r}_2}$: relative position vector of 2 particles

- Upon exchanging 2 particles, $\bar{r} \rightarrow -\bar{r}$

$\Rightarrow$ we require $\psi(-\bar{r}) = \psi(\bar{r})$

i.e., "like" even under $\boxed{\text{parity}}$ $\Rightarrow$ l must be even [$(-1)^l$ eigenvalue in general] $\Rightarrow$ total angular momentum j_{final} of final state $= l_{final}$ (since $s_{final} = 0$) $=$ even

... but then it cannot match initial angular momentum (spin) of 1

$\Rightarrow$ spin-1 particle cannot decay into 2 identical spin-0 particles