Lecture $\boxed{41}$, Dec. 9 (Wed.)

(continued from lecture 40 notes)

Outline for $\boxed{\text{today}}$ & Fri.

— "reversed" particle motion also valid

— intuition from wavefunction (time-reversal to do with "complex-conjugation")

— need "detour" on unitary operator (thus far) vs. anti-unitary (distinct from non-unitary!) in general (related to above complex conjugation)

— Back to time-reversal: must be anti-unitary (based on how Hamiltonian transforms)

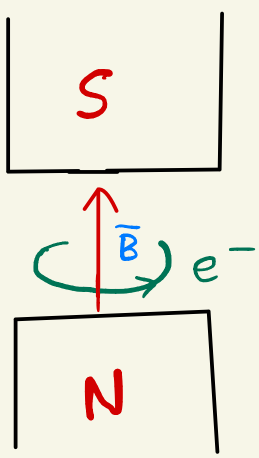
  — how other operators, expectation values, wavefunctions transform (matching intuition / educated guess of before)

— H is time-reversal invariant

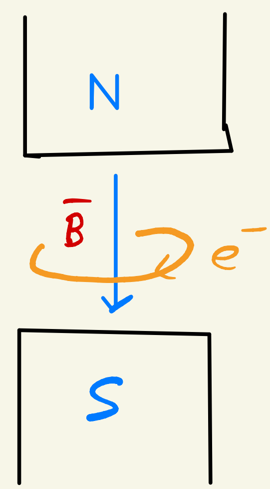
Reversed particle trajectory also makes sense $\left(\bar{p}\Big|_{t=0+\varepsilon} \longrightarrow -\bar{p}\Big|_{t=0-\varepsilon} \right)$

– If $\left[\bar{x}_{old}(t) = \right] \bar{x}(t)$ solves $m\ddot{\bar{x}} = -\bar{\nabla} V(\bar{x})$, then so does $\left[\bar{x}_{new}(t) = \right] \bar{x}(-t)$: no dissipation (so does not apply to object slowing down due to friction)

– Subtlety with $\bar{B}$: electron orbiting anti-clockwise : reversed not allowed?!



... but $\bar{B}$ due to moving charges (electric current): reverse that (hence $\bar{B}$), then valid!



Maxwell's and Lorentz force equations time-reversal invariant : $t \to -t$ if $E \to E$; $B \to -B$; $\rho \to \rho$; $\bar{j} \to -\bar{j}$; $\bar{v} \to -\bar{v}$

$$\boxed{\begin{array}{ll} \bar{\nabla}\cdot\bar{E} = 4\pi\rho & \bar{\nabla}\cdot\bar{B} = 0 \\ \bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t} & \bar{\nabla}\times\bar{B} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = 4\pi\bar{j} \end{array}}$$

&

$$\boxed{\bar{F} = e\bar{E} + \frac{e}{c}\bar{v}\times\bar{B}}$$

Onto QM: wavefunction behavior under time-reversal "easy": if

$$i\hbar \frac{\partial \psi}{\partial t}(x, t) = \left(-\frac{\hbar^2}{2m}\bar{\nabla}^2 + V\right)\psi(x,t),$$

then $\psi(x, -t)$ is not solution [cf. $\bar{x}(-t)$ is valid trajectory] due to

$$\partial/_{\partial t}\, \psi(x, -t) = -\left[\partial/_{\partial t}\, \psi(\bar{x}, t)\right]\Big|_{t\to -t}$$

(V real / H Hermitian crucial)

[cf. $\frac{d^2\bar{x}}{dt^2}$ ... classically, so $\bar{x}(-t)$ solution]

Remarkably, $\psi^*(\bar{x}, -t)$ solves [just complex conjugate above equation]

e.g., (sanity check) stationary state,

$$\psi(\bar{x}, t) = u_n(\bar{x})\exp(-iE_n t/\hbar) \Rightarrow$$

$$\psi^*(x, -t) = u_n^*(x)\exp(-iE_n t/\hbar)$$

["right" time-dependence]

$\Rightarrow$ Educated guess: time-reversal involves c.c.: (at $t=0$) wavefunction of time-reversed state is $\langle x|\alpha\rangle^*$

(vs. original $\langle x|\alpha\rangle$)

$\Rightarrow$ time-reversal "special" (cf. translation, rotation, even parity: $\psi(\bar{x},t) \to \psi(-\bar{x},t)$, but no c.c.

$\boxed{\text{Detour on/generalization of symmetry operations: anti-unitary operator}}$

– $\boxed{\text{So far}}$ (translation, rotation, parity...),

$|\tilde{\alpha}\rangle = U|\alpha\rangle$; $|\tilde{\beta}\rangle = U|\beta\rangle$, where

U is unitary $\Rightarrow$

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\overbrace{U^\dagger U}^{\mathbb{1}}|\alpha\rangle = \langle\beta|\alpha\rangle$$

$$\Rightarrow \int dx'\, \tilde{\psi}_\beta^*(x',0)\,\tilde{\psi}_\alpha(x',0) = \int dx'\, \psi_\beta^*(x',0)\,\psi_\alpha(x',0)$$

– However, enter time-reversal:

$$\tilde{\psi}_{\alpha,\beta}(x',0) = \psi_{\alpha,\beta}^*(x',0) \Rightarrow \langle\tilde{\beta}|\tilde{\alpha}\rangle$$

$$= \int dx'\, \psi_\beta(x',0)\,\psi_\alpha^*(x',0)$$

$$= \left[\int dx'\, \psi_\beta^*(x',0)\,\psi_\alpha(x',0)\right]^* = \left[\langle\beta|\alpha\rangle\right]^*$$

- So, "generalize" to $|\langle\tilde{\beta}|\tilde{\alpha}\rangle| = |\langle\beta|\alpha\rangle|$

so that (a) $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle$ (unitary operator)

or (b) $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*$ (anti-unitary operator such as time-reversal)

Anti-unitary operator (denoted by θ):

$|\tilde{\alpha}\rangle = \theta|\alpha\rangle$; $|\tilde{\beta}\rangle = \theta|\beta\rangle$ satisfies

(1). $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*$ as above and

(2). $\theta\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^*\,\theta|\alpha\rangle + c_2^*\,\theta|\beta\rangle$

(2^{nd} condition by itself is anti-linear)

$\boxed{\text{Claim}}$: anti-unitary operator, $\boxed{\theta = UK}$,

where U is unitary and K is "c.c."

operator : $Kc|\alpha\rangle = c^*K|\alpha\rangle$, but

basekets unchanged : $K|a'\rangle = |a'\rangle$:

reasonable, since $|a'\rangle \doteq \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \boxed{1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (purely real)

$$|\alpha\rangle = \sum_{a'} \langle a'|\alpha\rangle |a'\rangle \xrightarrow{K} |\tilde{\alpha}\rangle = \sum \langle a'|\alpha\rangle^* K |a'\rangle$$
$$= \sum \langle a'|\alpha\rangle^* |a'\rangle$$

Note: if S_z eigenkets $(|\pm\rangle)$ taken as base kets, then S_y eigenkets do change : $\frac{1}{\sqrt{2}}\left(|+\rangle \pm i|-\rangle\right)$

$\xrightarrow{K}$ $\frac{1}{\sqrt{2}}\left(|+\rangle \mp i|-\rangle\right)$ ... but if S_y eigenkets chosen as base kets, then no change due to K $\Rightarrow$ action of K basis-dependent

Property $\boxed{(A)}$ of $\theta = U K$: what's

basis independent — each depends on basis

U also basis dependent (return to it)

check $\theta = UK$ is anti-linear (condition #2) : $\overbrace{(U\,K)}^{\theta}\left(c_1|\alpha\rangle + c_2|\beta\rangle\right)$

$$= UK\, c_1|\alpha\rangle + UK\, c_2|\beta\rangle$$
$$= U c_1^* K|\alpha\rangle + U c_2^* K|\beta\rangle$$
$$= c_1^* (UK|\alpha\rangle) + c_2^* (UK|\beta\rangle) \quad \left(\begin{array}{c} U \text{ is} \\ \text{unitary} \end{array}\right)$$
$$= c_1^* \theta|\alpha\rangle + c_2^* \theta|\beta\rangle$$

For checking condition #1 for UK being anti-unitary, property $\boxed{(\beta)}$: act with $\theta = UK$ on ket from left (enough for computing inner product: no need to act with θ on bra from right): $|\tilde{\beta}\rangle = \sum_{a'} \langle a'|\beta\rangle\, U|a'\rangle$

$$\Rightarrow \quad \langle\tilde{\beta}| = \sum_{a'} \langle a'|\beta\rangle \langle a'|U$$

so that

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \sum_{a'}\sum_{a''} \langle a''|\beta\rangle \langle a''|\overbrace{U^\dagger U}|a'\rangle \langle\alpha|a'\rangle$$

(use $\langle a''|a'\rangle =$ $\qquad$) $= \sum_{a'} \langle\alpha|\underbrace{a'\rangle\langle a'|}\beta\rangle$

$=$