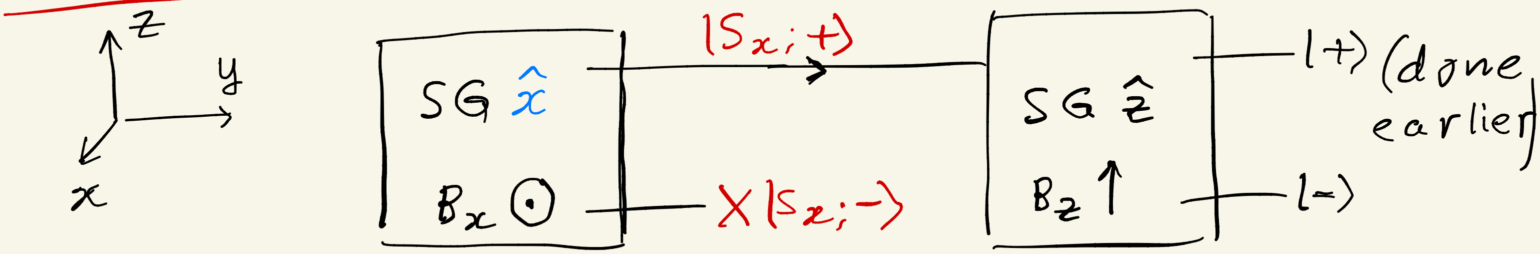
Last time: outcome of measurement on a system ("collapse" $|\alpha\rangle$ into $|a'\rangle$); probability to measure $a' = |\langle a'|\alpha\rangle|^2$ (if $\langle\alpha|\alpha\rangle = 1$); expectation value of an observable/operator

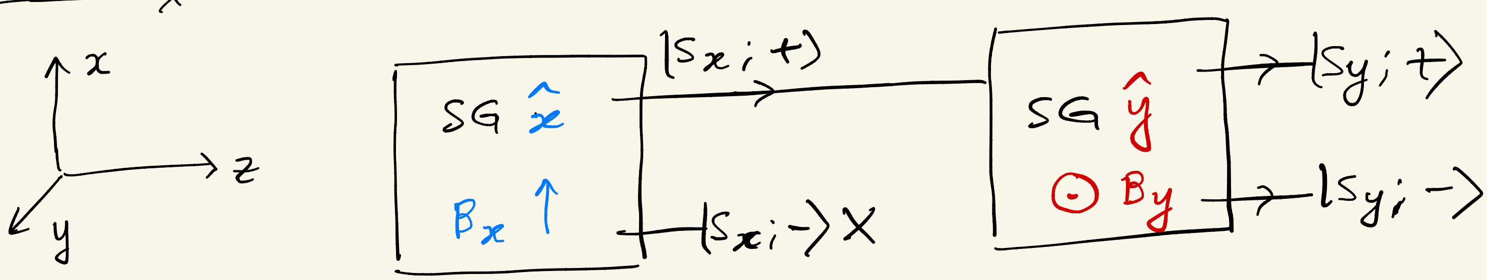
Lecture $\boxed{4}$ (Sept. $\boxed{9}$, Wednesday)

$\boxed{\text{Outline}}$ confirm earlier educated guess for $|S_{x,y};\pm\rangle$ in terms of $|S_z;\pm\rangle \equiv |\pm\rangle$ using measurement probabilities (will see that earlier discussion was a bit "incomplete")

- commuting operators (compatible observables) vs. not

Back to spin-½ system (SG experiment)





probability to get $|\pm\rangle$ in $|S_x;+\rangle$ equal ½ each $\Rightarrow$

$$|\langle +|S_x;+\rangle| = |\langle -|S_x;-\rangle| = 1/\sqrt{2} \quad \Rightarrow$$

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right)$$

$$|S_x;-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{+i\delta_1}|-\rangle\right)$$

construct $\boxed{S_x} = \frac{\hbar}{2}\left[\left(|S_x;+\rangle\langle S_x;+|\right) - \left(|S_x;-\rangle\langle S_x;-|\right)\right]$

$$= \hbar/2\left[e^{-i\delta_1}\left(|+\rangle\langle -|\right) + e^{i\delta_1}\left(|-\rangle\langle +|\right)\right]$$

similarly, $|S_y;\pm\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm e^{i\delta_2}|-\rangle\right)$

$$S_y = \hbar/2\left[e^{-i\delta_2}\left(|+\rangle\langle -|\right) + e^{i\delta_2}\left(|-\rangle\langle +|\right)\right]$$

$\rightarrow$ SG $\hat{x}$ & SG $\hat{y}$ ... $\Rightarrow$ $|\langle S_y;\pm|S_x;+\rangle| = 1/\sqrt{2}$
(2nd figure above) $= |\langle S_y;\pm|S_x;-\rangle|$

... $\boxed{\delta_2 - \delta_1 = \pm\pi/2}$ $\Rightarrow$ matrix elements of S_x & S_y not all real

$\rightarrow$ convention: $\boxed{\delta_1 = 0}$ & $\boxed{\delta_2 = \pi/2}$

$$S_y \doteq \hbar/2\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \hbar/2\,\sigma_2 \;;\; S_x \doteq \frac{\hbar}{2}\sigma_1$$

$\langle -|S_y|+\rangle$; $\langle -|S_y|-\rangle$

Also, $\boxed{\text{raising/lowering}}$ operators: $\boxed{S_\pm} = S_x \pm iS_y$

$S_+ = \hbar|+\rangle\langle -|$; $S_- = \hbar|-\rangle\langle +|$ $\Rightarrow$

$S_+|-\rangle = \hbar|+\rangle$; $S_+|+\rangle = 0$

– relations: $\boxed{[S_i, S_j] = i\epsilon_{ijk}S_k\hbar}$ (valid for any angular momentum)

$i = x, y, z$; $[S_i,S_j] = S_iS_j - S_jS_i$; $\epsilon_{123} = +1 = -\epsilon_{213}$ ...

$\boxed{\{S_i, S_j\} \equiv} \; S_iS_j + S_jS_i = \boxed{\frac{1}{2}\hbar^2 \delta_{ij}}$ (specific to spin-1/2)

$\boxed{S^2 \equiv \bar{S}\cdot\bar{S} = 3/4\, \hbar^2 \mathbb{1} \text{ (spin-1/2)}}$

$\boxed{[S^2, S_i] = 0 \text{ (general)}}$

HW 2.1 spin operator in general direction

Commuting operators / compatible observables

$\boxed{[A, B] = 0}$ e.g. S^2 & $S_{x,y,z}$

every eigenket of A is eigenket of B (form basis) (more practice with algebra in HW 2.4)

$$A\big(B|a'\rangle\big) = BA|a'\rangle = a'\big(B|a'\rangle\big)$$

$B|a'\rangle$ is eigenket of A with eigenvalue a'

$\Rightarrow B|a'\rangle \propto |a'\rangle$ (no degeneracy)

$\langle a'|$ $\equiv b'|a'\rangle \Rightarrow |a'\rangle$ is eigenket of B

$\langle a'|$

matrix representation: $|a'\rangle$ as basekets

($\Rightarrow$ A matrix is diagonal) : $\boxed{\langle a''|B|a'\rangle \text{ also diagonal}}$

Proof: $\langle a''|[A,B]|a'\rangle = 0$

$$= \langle a''|(AB - BA)|a'\rangle = \underbrace{(a'' - a')}_{\neq 0}\langle a''|B|a'\rangle$$

$$\Rightarrow \langle a''|B|a'\rangle = \delta_{a'a''}\langle a'|B|a'\rangle$$

$\langle a'|B|a'\rangle = b'\langle a'|a'\rangle \Rightarrow \boxed{b' = \langle a'|B|a'\rangle}$

$(\langle a'|a'\rangle = 1)$

$\boxed{\text{Measurement}}$ of A, followed by B ... $\boxed{\text{simple}}$ result

$$|\alpha\rangle \xrightarrow{\text{measure } A} |a', b'\rangle \xrightarrow{\text{measure } B} |a', b'\rangle$$

arbitrary

measured

measured

again, measure A

$|a', b'\rangle$

$\boxed{\text{B measurement does not erase A measurement information}}$

cf. $[A, B] \neq 0$, e.g., $[S_z, S_y] \neq 0$ (sequential SG, where S_y measurement "resets" S_z done before)