Lecture 38, Dec. 2 (Wed.)

Outline for today: Wigner-Eckart

(From Fri., onto ch. 4 on symmetries)

(3). Matrix elements of tensor operators (in angular momentum eigenstate basis)

(Wigner-Eckart theorem)

Motivation: interactions of atomic/nuclear states with EM field (e.g., chapter 5 in Phys 623 or AMO), i.e.,

transition amplitude ~

$$\Big\langle \underset{\substack{\text{(different)}\\ \text{angular}\\ \text{momentum}\\ \text{eigenket}}}{\text{final state}} \Big| \underbrace{\text{EM field}}_{\substack{\text{(sum of ?)}\\ \text{tensor}\\ \text{operators}}} \Big| \underbrace{\text{initial state}}_{\substack{\text{angular}\\ \text{momentum}\\ \text{eigenket}}} \Big\rangle$$

– Properties deduced from "kinematics or geometry" (rest depends on "non-angular" features, e.g., radial part of wavefunctions, thus needs case-by-case analysis)

(i). Kinematics-based selection rule:

$$\langle \alpha', j' m' | T_q^{(k)} | \alpha, j\, m \rangle$$

$\alpha \rightarrow$ e.g., $\boxed{\text{radial}}$ quantum number

$= 0$

unless $\boxed{m' = q + m}$

Proof: use $[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$ to write

$$\langle \alpha', j' m' | \overbrace{\left([J_z, T_q^{(k)}] - \hbar q T_q^{(k)} \right)}^{0} | \alpha, j\, m \rangle = 0$$

$$= \langle \alpha', j' m' | \overset{\boxed{m'\hbar}}{J_z} T_q^{(k)} - T_q^{(k)} \overset{\boxed{m\hbar}}{J_z} - \hbar q T_q^{(k)} | \alpha, j\, m \rangle$$

$$= \langle \alpha', j' m' | T_q^{(k)} | \alpha, j\, m \rangle \hbar (m' - m - q)$$

$\Rightarrow \langle \alpha, j\ m | T^{(k)} | \alpha, j m \rangle \neq 0$ only if $m' - m - q = 0$

[Not so simple for Cartesian tensor operator... have to decompose it into spherical tensor operators...]

(2). Wigner-Eckart theorem, giving another selection rule for matrix element of spherical tensor operator,

$$\underbrace{\langle \alpha', j' m' |}_{\text{different...}} T_q^{(k)} \underbrace{| \alpha, j m \rangle}_{\text{angular momentum eigenket}}$$

Intuition: even for given j, j', too many "orientations" to consider: m, m' (of initial & final states) and q (of tensor operator!

Instead of computing each case separately, it suffices to do it for one (suitably chosen) easier to compute?) set of orientations; then obtain rest (for other values of m, m', q) in terms of "seed / parent" using "geometry"

Claim (proof in Sakurai):

$$\langle \alpha', j' m' | T_q^{(k)} | \alpha, j m \rangle = \underbrace{\langle j k ; m q | j k ; j' m' \rangle}_{\text{CG coefficient}}$$

$$\times \langle \alpha', j \| T^{(k)} \| \alpha, j \rangle \big/ \sqrt{(2j+1)}$$

notation: independent of m, m' & q (orientations relative to z-axis)

"Realizes" intuition: matrix element of (spherical) tensor operator "factorizes": it is product of

(1). CG coefficient as relevant for adding $j\, m$ (initial state) & $k\, q$ (of operator) to get $j' m'$ (final state): for given j, j', this part depends only on orientations (geometry), not on radial part (α) of initial/final state wavefunctions ... or other than angular features of operator

(2). "dynamical" factor: $\langle \alpha' j' \| T^{(k)} \| \alpha, j \rangle$ which does not depend on m, q & m' (geometry), but contains information about radial part of wavefunctions (and non-angular details of tensor operator).

$\Rightarrow$ In order to compute matrix elements for various m, q & m',

step (a): calculate for one choice of m, m' & q (involved: radial integrals... but no need to do it again!)

step (b): get other matrix elements via CG coefficients (kinematical factor)

— $\boxed{CG}$ coefficient non-vanishing (hence selection rule) only if

(i) $m' = m + q$ (same as earlier)

(ii) $\boxed{|j-k| \leq j' \leq (j+k)}$

Examples (more in HW 10):

(I). Scalar operator: $T^{\overset{k}{(0)}}_{\underset{q}{0}} \equiv S \Rightarrow$

$$\langle \alpha', j' m' | S | \alpha, j m \rangle = \frac{\langle \alpha' j \| S \| \alpha, j \rangle}{\sqrt{(2j+1)}}$$

$$\times \; \delta_{jj'} \, \delta_{mm'}$$

$$|j - \underset{k}{0}| \leq j' \leq j + \underset{k}{0} \qquad m' = m +$$

... as expected based on operator carrying "no angular momentum" ($k = 0$; $q = 0$), so can't change j or m of initial state

(II). Vector operator / spherical rank 1 tensor operator:

$V_{i=1,2,3}$ (or x, y, z) "re-arranged" as $V_{q=0,\pm 1}$ $\Rightarrow$

$$\boxed{\Delta m = m' - m = \pm 1, 0 \; (= q)} \; \&$$

$$\overbrace{|j - \underset{K}{\underset{\uparrow}{1}}|}^{(j-1) \text{ for } j \geq 1} \leq j' \leq j + 1 \leftarrow K$$

$$\Rightarrow -1 \leq \underbrace{j' - j}_{\Delta j} \leq +1$$

i.e., $\Delta j = \begin{cases} \pm 1 \\ 0 \end{cases}$ for $j \geq 1$

However, for $j = 0$ (initial state rotation-invariant), we get

$$|\underset{\substack{\nearrow \\ j}}{0} - \underset{\substack{\uparrow \\ k}}{1}| \leq j' \leq |\underset{\substack{\uparrow \\ j}}{0} + \underset{\substack{\uparrow \\ k}}{1}|$$

$\Rightarrow$ only $j' = 1$ allowed:

$0 \longrightarrow 0$ transition forbidden

(as expected based on operator having non-zero ("one") angular momentum, thus must give final state with non-zero (one) angular momentum when acting on initial state with vanishing angular momentum

— Restricting to $j' = j$ (initial & final j same) and vector operator, WE becomes "projection theorem"

(see Sakurai)