Lecture $\boxed{37}$, Nov. 30 (Mon.):

(continued from lecture 36 notes)

Outline for today (& Wed.): after

(1). Vectors, go to (2). Tensors: Cartesian defined as "generalization" of (Cartesian) vector

(a). Cartesian tensor (obtained by "product" of -Cartesian- vectors) decomposes into irreducible spherical tensors, defined as generalization of spherical harmonics

(b). Product of spherical tensors (similar to product of Cartesian vectors)

(3). Martix elements of tensor operators (Wigner-Eckart theorem), e.g., EM transition amplitude (chapter 5: Phys 623 or AMO)

(a) Product of Cartesian vectors (Cartesian tensor) motivates spherical tensor

Dyadic : $T_{ij} \equiv U_i V_j$, where $\bar{U}, \bar{V}$ are vectors (numbers for now: $i,j = 1,2,3$), transforming as (based on $\bar{U}, \bar{V}$ properties) rank two tensor:

$$T_{ij} \longrightarrow \sum_{i'} \sum_{j'} R_{ii'} R_{jj'} T_{i'j'},$$

given $U_i \longrightarrow \sum_{i'} R_{ii'} U_{i'}$; $V_j \rightarrow \sum_{j'} R_{jj'} V_{j'}$

– T has 9 components

– "disadvantage": T_{ij} is reducible: it splits into multiple "objects" transforming differently & independently ("within themselves") under rotations:

$$U_i V_j = \underbrace{\frac{\bar{U}\cdot\bar{V}}{3}\delta_{ij}}_{\text{trace}} + \underbrace{\left(\frac{U_i V_j - U_j V_i}{2}\right)}_{\text{anti-symmetric}}$$

(check: 3 RHS terms add up to LHS)

$$+ \left[(U_i V_j + U_j V_i)/2 - \frac{\bar{U}\cdot\bar{V}}{3}\delta_{ij}\right],$$ traceless, symmetric

– "Trace" component (1) under rotations is invariant

– Anti-symmetric (number of components: $^3C_2$: off-diagonal = 3) transforms as vector : $T_{ij}^{\text{anti-symmetric}} \propto \varepsilon_{ijk} (\bar{U} \times \bar{V})_k$

– Traceless, symmetric : number of components = $^3C_2$ (off-diagonal) + 3 (diagonal) -1 (traceless) = 5, i.e., sanity check : total 9 to begin with

$-$ 1 (trace)

$-$ 3 (antisymmetric)

– Note : dyadic breaks up into objects with d.o.f. "corresponding to" angular momenta $\ell = 0, 1, 2$, "like" spherical harmonics ("tensors"):

$$Y_\ell^{m} (\theta, \phi) \quad m \in -\ell, -\ell+1 \ldots \ell-1, \ell \; (= 2\ell+1 \text{ values})$$

... this is decomposition of Cartesian tensor into irreducible spherical tensors

We'll $\boxed{\text{define}}$ spherical tensor operators in terms of transformation under rotations, or (equivalently) commutation relations with angular momentum operators

... just like with vectors

– Just like with vector operators, start classicaly, i.e., with numbers; here, with spherical harmonics (motivated by above)

– So, warm-up with example/$\boxed{\text{one}}$ way to build spherical tensor (numbers to begin with → operators later):

$$Y_\ell^m(\underbrace{\theta, \phi}_{\hat{n}}) \xrightarrow{\hat{n} \to \bar{V} \text{ (vector)}} \boxed{Y_{\ell=k}^{m=q}(\bar{V})} \equiv T_q^{(k)}$$

k ← rank ("like" ℓ)

q ← "magnetic" quantum number ("like" m)

spherical tensor of rank $\boxed{k}$, with $(2k+1)$ components (values of q)

... still a bit abstract, so choose

$k = 1$, i.e., $l = 1$ spherical harmonics,

with $\cos\theta = \frac{z}{r} = (\hat{n})_z \longrightarrow V_z$

and $\sin\theta\, e^{\pm i\phi} = \frac{x \pm iy}{r} \longrightarrow V_x \pm V_y i$

so that

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \longrightarrow \underset{\text{"z"-component}}{T_0^{(1)}}{}^{\leftarrow \text{rank}} = \frac{3}{4\pi} V_0 (= V_z)$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{4\pi}} \frac{\sin\theta\, e^{\pm i\phi}}{\sqrt{2}} \longrightarrow T_{\pm 1}^{(1)}$$

[Note: rank 1 spherical tensor is just Cartesian "re-written"]

$$= \sqrt{\frac{3}{4\pi}} \underbrace{\left[\mp \frac{(V_x \pm i V_y)}{\sqrt{2}} \right]}_{V_{\pm 1}}$$

Similarly, rank 2 [$T_q^{(2)}$ is 1 part of T_{ij}]

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \left(\frac{x \pm iy}{r} \right)^2 \longrightarrow T_{\pm 2}^{(2)} = \sqrt{\frac{15}{32\pi}} \underbrace{(V_x + i V_y)^2}_{2(V_{\pm 1})^2}$$

check

... again, one path to T's (not yet formal definition or operators)

Advantage of $T_q^{(k)}$ (spherical) [over T_{ij} (Cartesian)] : irreducible (like Y_ℓ^m's)

... leads us to transformation under rotations : "inspired" by how $Y_\ell^m(\theta, \phi)$ transform, "translating" that to $Y_{\ell(=k)}^{m(=q)}(\bar{V})$ ... finally, $T_q^{(k)}$ operator

[again, similar prescription as for vectors classical/number $\longrightarrow$ operator]

- Start with $|\hat{n}\rangle \longrightarrow \mathcal{D}(R)|\hat{n}\rangle$
$= |\hat{n}'\rangle$
(rotated eigenket)

- Use :

note

$$\langle\hat{n}|\mathcal{D}(R^{-1})|\ell, m\rangle = \sum_{m'} \langle\hat{n}|\ell, m'\rangle \underbrace{\mathcal{D}_{m'm}^{(\ell)}(R^{-1})}_{\text{matrix element of } \mathcal{D}(R^{-1})}$$

[Use $\langle \hat{n} | l, m \rangle = Y_l^m(\underbrace{\theta, \phi}_{\hat{n}})$ on

both sides and $\langle \hat{n}' | = \langle \hat{n} | [\mathcal{D}(R)]$ (on LHS)

$= \langle \hat{n} | \mathcal{D}(R^{-1})$, since $\mathcal{D}$ is

unitary : $\mathcal{D}(R^{-1}) = [\mathcal{D}(R)]^{\dagger}$]

$$Y_l^m(\hat{n}') = \sum_{m'} Y_l^{m'}(\hat{n}) \, \mathcal{D}^{(l)}_{m'm}(R^{-1})$$

rotated — original

i.e., how $Y_l^m(\hat{n})$ transforms ...

... with $\hat{n} \to \bar{V}$, $Y_l^m(\bar{V})$ rotate similarly...

$\Rightarrow$ promoting $\bar{V}$, hence $Y_l^m(\bar{V})$ also, to operator, it is reasonable to require

$$\underbrace{\mathcal{D}^{\dagger}(R) \, Y_l^m(\bar{V}) \, \mathcal{D}(R)}_{\text{transformation of operator ("like" } \bar{V} \text{ itself)}} = \sum_{m'} Y_l^{m'}(\bar{V}) \, \mathcal{D}^{(l)*}_{mm'}(R)$$

use $\mathcal{D}(R^{-1}) = [\mathcal{D}(R)]^{\dagger}$

$\Rightarrow$ $\boxed{\text{in general}}$, spherical tensor operator in QM defined as (via fundamental / rotational property)

$$\mathcal{D}^{\dagger}(R)\, T_q^{(k)}\, \mathcal{D}(R) = \sum_{q'=}^{q'=} \mathcal{D}^{(k)*}_{qq'}(R)\, T^{(k)}_{q'}$$

rank

(2k+1) components

$$= \sum_{q'} \mathcal{D}^{(k)}_{q'q}(R)\, T^{(k)}_{q'}$$

... "forgetting" its (possible) "origin" as $Y_{l=k}^{m=q}(\bar{V})$ (in motivating example), e.g., $(U_x + iU_y)(V_x + iV_y)$ $(U \neq V)$ is $T_2^{(2)}$, even if unlike $(V_x + iV_y)^2$ (of before), it's not of form $Y_2^2(\bar{V})$ [again, $Y_l^m(\bar{V})$ is one option to get T...]

— Equivalently, $T_q^{(k)}$ defined using its commutation relations with angular momentum ...

... follows from infinitesimal version of above:

$$\underbrace{\left(1 + i\frac{\bar{J}\cdot\hat{n}}{\hbar}\varepsilon\right)}_{\mathcal{D}^{\dagger}} T_q^{(k)} \underbrace{\left(1 - i\frac{\bar{J}\cdot\hat{n}}{\hbar}\varepsilon\right)}_{\mathcal{D}} = \sum_{q'=-k}^{q'=k} T_{q'}^{(k)} \underbrace{\langle k q' | \left(1 + i\frac{\bar{J}\cdot\hat{n}\varepsilon}{\hbar}\right) | k q\rangle}_{\mathcal{D}^{(k)}_{qq'}}$$

($|k q\rangle$ are angular momentum eigenkets)

$$\Rightarrow \left[\bar{J}\cdot\hat{n}, T_q^{(k)}\right] = \sum_{q'} T_{q'}^{(k)} \langle k q' | \bar{J}\cdot\hat{n} | k q\rangle$$

In order to "collapse" $\sum_{q'}$, choose

(a). $\hat{n} = \hat{z}$ so that on RHS, $\langle k q' | J_z | k q\rangle = q\,\delta_{q'q}\hbar$

giving $\left[J_z, T_q^{(k)}\right] = \hbar q\, T_q^{(k)}$

(b). $\hat{n} = \hat{x} \pm i\hat{y}$ and

$$\langle k, q' | J_{\pm} | k, q\rangle = \delta_{q', q\pm 1}\sqrt{(k \mp q)(k \pm q + 1)}\,\hbar$$

to get $\boxed{\left[J_{\pm}, T_q^{(k)}\right] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\, T^{k}_{q\pm 1}}$

special case: $k = 1$ ... reduces to vector with $[V_i, J_j] = i\varepsilon_{ijk}\hbar V_k$

Product of spherical tensors (cf. earlier, product of Cartesian vectors gave scalar, vectors/anti-symmetric tensor & traceless symmetric tensor), e.g., "multiply" two spherical rank 1 tensors ["same" as Cartesian vectors]

recalling $V_{x,y,z}$ (Cartesian) $\longrightarrow$ $V_0 (= V_z)$ & $V_{\pm 1} \left[= \mp \frac{(V_x \pm i V_y)}{\sqrt{2}} \right]$

(V_q, with $q = 0, \pm 1$ are components of spherical tensor)

$$T_0^{(0)} = -\frac{\bar{U} \cdot \bar{V}}{3} = \frac{(U_{+1} V_{-1} + U_{-1} V_{+1} - U_0 V_0)}{3}$$

$$T_q^{(1)} = (\bar{U} \times \bar{V})_q / (i\sqrt{2})$$

$$T_{\pm 2}^{(2)} = U_{\pm 1} V_{\pm 1} \; ; \; T_0^{(2)} = \frac{(U_{+1} V_{-1} + U_{-1} V_{+1} + 2 U_0 V_0)}{\sqrt{6}}$$

$$T_{\pm 1}^{(2)} = (U_{\pm 1} V_0 + U_0 V_{\pm 1}) / \sqrt{2}$$

Check compare above $T_q^{(k)}$'s (product of $\bar{V}$ or $T^{(1)}$'s) with $Y_\ell^m \underbrace{(\theta, \phi)}_{\hat{n}}$ (and $\hat{n} \rightarrow \bar{V}$; $V_i \rightarrow V_q$), e.g., (earlier way)

$$T^{(2)}_{\pm 2} \ (\text{for } U = V) = V_{\pm 1}^2 = \left(\frac{V_x \pm i V_y}{\sqrt{2}}\right)^2$$

$$\propto \boxed{Y_2^{\pm 2}(\bar{V})}, \text{ since } Y_2^{\pm 2}(\theta,\phi) \propto (\sin\theta e^{\pm i\phi})^2$$

$$= [\underbrace{(x \pm i y)/r}_{\to V_x \pm i V_y}]^2$$

Similarly, $\underline{\text{above}}$ $\boxed{T_0^{(2)}}$ (with

$$U = \bar{V}) = [-(V_+ V_-)^2 + V_0^2]\, 2/\sqrt{6}$$

$$= \left[-\frac{(V_x^2 + V_y^2)}{2} + V_z^2\right] 2/\sqrt{6}$$

$$\propto Y_2^0(\bar{V}), \text{ since } Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(\frac{3z^2 - r^2}{r^2}\right)$$

$$\propto [\underbrace{2 z^2}_{\to V_z^2} - \underbrace{(x^2 + y^2)}_{\to V_x^2 + V_y^2}]$$

More systematically, use theorem: if $X^{(k_1)}_{q_1}$ & $Z^{(k_2)}_{q_2}$ are irreducible tensors of rank $k_{1,2}$, then

$$\boxed{T^{(k)}_q = \sum_{q_1}\sum_{q_2} \underbrace{\langle k_1 k_2; q_1 q_2 | k_1 k_2; k q \rangle}_{\text{CG coefficient}} \times X^{(k_1)}_{q_1} Z^{(k_2)}_{q_2}}$$

is spherical tensor of rank k.

"Intuition" (proof in Sakurai): "like" adding two angular momenta to generate other values (hence same CG coefficients appear) $\Rightarrow$ it's "like" $\boxed{T^{(k)}_q}$ has "angular momentum k" (with z-component being q), even though it's operator (not state)