Lecture $\boxed{36}$, Nov. 23 (Mon.)

(continued from lecture 35 notes)

$\boxed{\text{Tensor operators (sec 3.11)}}$

- So far, how states transform under rotations: next, rotational properties of operators
- Motivation: EM field operator causes transitions between atomic states, so rotational properties of "ingredients" (EM field operator & states) of transition amplitude elucidates its structure (gives "selection rules")
- $\boxed{\text{Outline}}$ for next few lectures

(1). Vector operators (simple), e.g., $\vec{x}$, $\vec{p}$; $\vec{S}$, $\vec{L}$ (latter generating rotations)

(2). Onto more complicated tensor operators ("generalizing" vector)

e.g., in 4d, $\underbrace{F_{\mu\nu}}_{\text{antisymmetric}}$ (EM field) = $\bar{E}$, $\bar{B}$ $\Rightarrow$ only 6 components $(= {}^4C_2)$

or $g_{\mu\nu}$ (metric tensor)

or generator of Lorentz transformations (boosts & rotations)

(a). Cartesian tensor (obtained by product of vectors) decomposes into irreducible spherical tensors

(b). Product of spherical tensors

(3). Martix elements of tensor operators (Wigner-Eckart theorem), e.g., EM transition amplitude (chapter 5: Phys 623)

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x

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(1). Vector operators $[$e.g., $\bar{x}$, $\bar{p}$, $\bar{L}]$: rotations generated by $\bar{J}$, so how operator, A transforms under rotations "defined/dictated" by $[A, \bar{J}]$ ... kind of "like" time-evolution of A (in H-picture) given by $[A, H]$, where Hamiltonian generates time translation

- Classically, vector (3 components) transform as $\underset{\text{number}}{V_i} \to \sum \underbrace{R_{ij}}_{3\times 3 \text{ orthogonal}} V_j$ under rotations

- So, by Ehrenfest theorem, expectation value of operator in QM transforms likewise: using $|\alpha\rangle \to \mathcal{D}(R)|\alpha\rangle$, we get $\langle\alpha|\underset{\text{operator}}{V_i}|\alpha\rangle \to \langle\alpha|\mathcal{D}^\dagger(R)V_i\mathcal{D}(R)|\alpha\rangle$

$$\underset{=}{\boxed{\text{require}}} \sum_{ij} R_{ij}\langle\alpha|V_j|\alpha\rangle$$

for any ket $\Rightarrow$ at operator-level,

$$\mathcal{D}^\dagger(R)\, V_i\, \mathcal{D}(R) = \sum_{ij} R_{ij} V_j$$

- Go to infinitesimal version:

$$\mathcal{D}(R) \simeq 1 - i\varepsilon\, \vec{J}\cdot\hat{n}/\hbar \Rightarrow$$

$$\mathcal{D}^\dagger(R)V_i\mathcal{D}(R) \simeq \left(1 + i\varepsilon\, \vec{J}\cdot\hat{n}/\hbar\right) V_i \left(1 - i\varepsilon\frac{\vec{J}\cdot\hat{n}}{\hbar}\right) \Rightarrow$$

$$V_i + \frac{\varepsilon}{i\hbar}\left[V_i, \vec{J}\cdot\hat{n}\right] = \sum_j R_{ij}(\hat{n};\varepsilon) V_j$$

(drop ε^2)

– choose $\hat{n} = \hat{z}$ so that

$$R(\hat{z};\varepsilon) = \begin{pmatrix} 1 & -\varepsilon & 0 \\ +\varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$i = 1$: $V_x + \frac{\varepsilon}{i\hbar}\left[V_x, J_z\right] = V_x - \varepsilon V_y$

$i = 2$: $V_y + \frac{\varepsilon}{i\hbar}\left[V_y, J_z\right] = \varepsilon V_x + V_y$

$i = 3$: $V_z + \frac{\varepsilon}{i\hbar}\left[V_z, J_z\right] = V_z$

$$\Rightarrow \boxed{\left[V_i, J_j\right] = i\varepsilon_{ijk}\hbar V_k}$$

i.e., "same" as J_i

– Finite rotation of V given by

$\exp\left(\frac{iJ_j\phi}{\hbar}\right) V_i \exp\left(-iJ_j\phi/\hbar\right)$

... use Baker-Hausdorff formula (like done for $\bar{S}$ operator) to get

$$\exp\left(i J_z \phi/\hbar\right) V_x \exp\left(-i J_z \phi/\hbar\right)$$

$$= V_x \cos\phi - V_y \sin\phi$$

$\Rightarrow$ $\boxed{\text{define}}$ vector operator by

$$[V_i, J_j] = i\,\varepsilon_{ijk}\,\hbar\, V_k$$

$\bar{J}$ itself is vector ; similarly $\bar{x}$, $\bar{p}$

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(2) ... generalize to tensor (rank n, $\boxed{\text{Cartesian}}$), $\boxed{T_{ijk\ldots}}$:

$$\boxed{T_{ijk} \to \sum_{i'} \sum_{j'} \sum_{k'} \ldots R_{ii'} R_{jj'} R_{kk'} \ldots T_{i'j'k'\ldots}}$$

(rotate each index of T like V...)