Lecture $\boxed{35}$, Nov. 20 (Fri.);

continued from lecture 34 notes

- Schwinger's oscillator model (related to angular momentum): theory, applications
- Next topic: tensor operators (sec. 3.11)

Physical interpretation of pair of uncoupled oscillators satisfying angular momentum commutation relations:

Claim/proposal: "identify" one "quantum/unit" of "$\pm$" type oscillator with one spin-½ particle with spin-$\frac{\text{up}}{\text{down}}$ (along $\pm z$)

$\Rightarrow n_\pm$ is $\boxed{\text{number}}$ of spin-up/down

check: actions of $J^{osc}_{\pm, z}$ on $|n_+, n_-\rangle$ are like those of "usual" $J_{\pm, z}$:

$$J^{osc}_+ |n_+, n_-\rangle = \hbar\, a_+^\dagger a_- |n_+, n_-\rangle$$

$$= \hbar \sqrt{n_-(n_+ + 1)}\, |n_+ + 1, n_- - 1\rangle$$

(creates one unit of + type oscillator, and annihilates one unit of − type)

$\longleftrightarrow$ (corresponds to) creates one spin-up and removes one spin-down

$\Rightarrow$ (usual) J_z increases by $\hbar$ $\longleftrightarrow$ $\Rightarrow$

expect $J_z^{\boxed{osc}}$ to also go up by $\hbar$ ...

Indeed, $J_z^{osc}\left(J_+^{osc}|n_+, n_-\rangle\right)$

$$= \boxed{\sqrt{n_-(n_+ +1)}\,\hbar\, J_z^{osc} |n_+ + 1, n_- - 1\rangle}$$

$$= \sqrt{n_-(n_+ +1)}\,\hbar\,\frac{\hbar}{2}\left[(n_+ + 1) - (n_- - 1)\right]|n_+ + 1, n_- - 1\rangle$$

$$= \left[\frac{\hbar}{2}(n_+ - n_-) + \hbar\right]\left(J_+^{osc}|n_+, n_-\rangle\right)$$

$\Rightarrow$ $\left(J_+^{osc}|n_+, n_-\rangle\right)$ is eigenket of J_z^{osc} with eigenvalue $\left[\hbar/2\,(n_+ - n_-) + \hbar\right]$ ("new")

vs. $J_z^{osc}|n_+, n_-\rangle = \frac{\hbar}{2}\boxed{(n_+ - n_-)}|n_+, n_-\rangle$ "old" eigenvalue

i.e., J_z^{osc} raised by $\hbar$ due to acting J_+^{osc}

Similarly, J_-^{osc} lowers J_z^{osc} ...

$$J_-^{osc}|n_+, n_-\rangle = \hbar\, a_-^\dagger a_+ |n_+, n_-\rangle$$
$$= \sqrt{n_+(n_-+1)}\ \hbar\, |n_+-1, n_-+1\rangle$$

However, $J_\pm^{osc}$ do not modify total number of oscillators, $N = n_+ + n_-$, just "shuffled" between $\pm$, thus $|\vec{J}^{osc}|^2$ is same $\leftrightarrow$ usual $J_\pm$ not affecting $|\vec{J}|^2$

Relatedly, $J_z^{osc} = \hbar/2\,(N_+ - N_-)$ counts (upto $\frac{\hbar}{2}$) difference of $\pm$ type oscillators

$\leftrightarrow$ (number of spin-up) $-$ (number of spin-down)

= usual J_z (upto $\hbar/2$)

$\Rightarrow$ even physically $\vec{J}^{osc}$ like angular momentum (upon identifying $\pm$ oscillator with spin-up/down)

$\Rightarrow$ can map the 2 further; use oscillator realization to deduce formula for angular momentum, e.g., compare $J_\pm^{osc}$ actions on $|n_+, n_-\rangle$ above to usual $J_\pm$ of before:

$$J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\ \hbar\, |j, m \pm 1\rangle$$

$$\Rightarrow \boxed{n_\pm \leftrightarrow (j \pm m)} \quad \left(\begin{array}{c}\text{mapping } J_{\pm} \text{ on} \\ \text{2 sides}\end{array}\right)$$

Sanity check: $\boxed{|\vec{J}^{\,osc}|^2}$ eigenvalue =

$$\hbar^2/_2 \, N\left(\frac{N}{2}+1\right) = \frac{\hbar^2}{2}(n_+ + n_-)\left(\frac{n_+ + n_-}{2}+1\right)$$

$$\leftrightarrow \hbar^2/_2 \, (j+m+j-m)\left(\frac{j+m+j-m}{2}+1\right)$$

$= \hbar^2 j(j+1) =$ eigenvalue of usual $|\vec{J}|^2$

i.e., $(n_+ + n_-) \leftrightarrow 2j$

Similarly $\boxed{J_z^{osc}}$ eigenvalue $= \frac{\hbar}{2}(n_+ - n_-)$

$$\leftrightarrow \hbar/_2 \left[j+m-(j-m)\right] = m\hbar$$

$=$ eigenvalue of usual J_z

i.e., $(n_+ - n_-)/_2 \leftrightarrow m$

Again, J_+^{osc} changes $\overbrace{n_\pm \text{ to } n_\pm \pm 1}^{(n_+ + n_-)\text{ same}}$

$\leftrightarrow j$ unchanged, m up by 1, i.e., "like" usual J_-

J_-^{osc} does $n_\pm$ to $n_\pm \mp 1$ $\left(\begin{smallmatrix} n_+ + n_- \\ \text{unchanged}\end{smallmatrix}\right)$ $\longleftrightarrow$

j unchanged, m down by 1, i.e.,

same as usual J_-

(all expected based on $\vec{J}^{osc}$ satisfying angular momentum commutation relations)

oscillators

Eigenket of $N_\pm$, $|n_+, n_-\rangle = \dfrac{(a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-}}{\sqrt{n_+!}\,\sqrt{n_-!}}$ acting on $|0,0\rangle$

$\longleftrightarrow |j, m\rangle$

"=" (can be thought of as)

spin-½

$$\frac{(a_+^\dagger)^{j+m} (a_-^\dagger)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} |0\rangle$$

, where $|0\rangle$ has zero angular momentum: no spin-½ particles

$\Rightarrow$ any large angular momentum (j, m: j = integer or half-integer) "made of" fundamental $2j$ spin-½ particles:

$j \pm m$ with spin up/down, so

$J_z = \hbar/2 \left[(j+m) - (j-m)\right] = m\hbar$

e.g., $\boxed{m = +j}$ (largest J_z eigenvalue for given j): $|j, j\rangle = (a_+^\dagger)^{2j}|0\rangle / \sqrt{(2j)!}$ is all $2j$ spins-up

... as far as rotations are concerned

Note: cannot take it literally, e.g., really adding angular momenta of $2j$ spin-½ particles gives net angular momentum: j (totally symmetric), $j-1$, $j-2$ ...; e.g., net spin $S = 1, 0$ from 2 spin-½ ...
($S = 1$, triplet is symmetric, vs. $S = 0$, singlet is anti-symmetric)

... but combining $2j$ oscillators "identified" as spin-½ gives only angular momentum j (totally symmetric combination, "as if" constituent spin-½ are "bosons"...

... as expected: they are oscillators)

– Relatedly, "isospin" in nuclear physics: proton/neutron "like" spin up/down or ± type oscillator: J_+^{osc} (or usual J_+)
$\longleftrightarrow$ isospin ladder operator (T_+ or I_+):

create proton & destroy neutron (isospin up/down), increases "z" (or 3rd) component of isospin (T_3 or I_3) by 1 ... relatedly, T_3 (or I_3) counts difference in number of protons & neutrons ("like" J_z^{osc} or usual J_z)

Summary: oscillators do not really "have" angular momenta, but it's just that algebra "like" angular momentum : $|n_+, n_-\rangle \leftrightarrow |j, m\rangle$,

number of + type   − type

with $\left(\frac{n_+ + n_-}{2}\right) \leftrightarrow j$ & $\left(\frac{n_+ - n_-}{2}\right) \leftrightarrow m$

...so can borrow results from oscillators (easier to work with) for angular momentum

Application of Schwinger's model: calculating rotation matrices

- Rotation operators via Wigner's method, e.g., $d^{(j=1/2,1)}$ earlier (lecture & HW

... but quite involved for higher j

- Easier way using Schwinger's model, since $a^{\dagger}, a$ (oscillator ladder operators) simpler to manipulate

- Recall goal: $\underbrace{\mathcal{D}}_{\text{rotation operator}}$ $\underbrace{|j,m\rangle}_{\text{angular momentum eigenket}}$ (arbitrary state expressed as sum)

$$= \overbrace{\sum_{m'} |j,m'\rangle\langle j,m'|}^{1} \underbrace{\mathcal{D}\, |j,m\rangle}_{\text{want this: } \mathcal{D}^{(j)}_{m'm}}$$

(simple) $e^{-im'\alpha}$ (simple) $e^{-im\gamma}$

$$\mathcal{D}^{(j)}_{m'm} = \langle j,m'| \exp\left(\frac{-iJ_z\alpha}{\hbar}\right) \exp\left(\frac{-iJ_y\beta}{\hbar}\right) \exp\left(\frac{-iJ_z\gamma}{\hbar}\right) |j,m\rangle$$

since $\mathcal{D} = \mathcal{D}(\alpha)\,\mathcal{D}(\beta)\,\mathcal{D}(\gamma)$ (Euler angles)

So, focus on middle (y), non-trivial, rotation: $\mathcal{D}^{(j)}_{m'm}(\alpha=0, \beta, \gamma=0)$ denoted by

$$\boxed{d^{(j)}_{m'm}(\beta) \equiv \langle j, m' | \underbrace{\exp\left(\frac{-i J_y \beta}{\hbar}\right)}_{\mathcal{D}(\alpha=0,\beta,\gamma=0)} | j, m \rangle}$$

i.e., "restrict" to

$$\underbrace{\mathcal{D}(\alpha=0,\beta,\gamma=0)}_{\substack{\text{(relevant)}\\ \text{rotation operator}\\ \text{(only about y-axis)}}} |j,m\rangle = \sum_{m'} |j,m'\rangle \underbrace{d^{(j)}_{m'm}(\beta)}_{\text{matrix element}} \quad \ldots(1)$$

(Recall: matrix element is amplitude to "find" $|j,m'\rangle$ in rotated $|j,m\rangle$)

...this is angular momentum "language"

$\boxed{\text{Big picture/strategy}}$ for getting $d^{(j)}_{m'm}$

Next, "switch" to "oscillator-inspired" picture on both sides of (1): $|j,m\rangle$ "constituted" by $2j$ spin-$1/2$, with $(j \pm m)$ spin-up/down

$\Rightarrow$ powers of $a^\dagger_\pm$ (creation operators of spin-up/down or $\pm$-type oscillators) on each side: "match" to obtain $d^{(j)}_{m'm}$

$\boxed{\text{RHS}}$ of (1) becomes

$$\sum_{m'} d^{(j)}_{m'm}(\beta) \underbrace{\frac{(a^\dagger_+)^{j+m'}(a^\dagger_-)^{j-m'}|0\rangle}{\sqrt{(j+m')!\,(j-m')!}}}_{|j,m'\rangle}$$

(easy)

$\boxed{\text{LHS}}$ of (1) in general, i.e, not just rotation about y-axis:

$$\mathcal{D}(R)|j,m\rangle = \underbrace{\mathcal{D}(R)}_{\text{use "}\mathcal{D}\text{"}} \underbrace{\frac{(a^\dagger_+)^{j+m}(a^\dagger_-)^{j-m}|0\rangle}{\sqrt{(j+m)!\,(j-m)!}}}_{|j,m\rangle}$$

$$\propto \mathcal{D}\, a^\dagger_+ \underbrace{\mathcal{D}^{-1}\,\mathcal{D}}_{\mathbb{1}}\, a^\dagger_+ \mathcal{D}^{-1} \mathcal{D} \ldots \mathcal{D}^{-1} \mathcal{D}\, a^\dagger_+ \ldots \Big\} \text{ from } \mathcal{D}(a^\dagger_+)^{(j+m)}$$

$$\underbrace{\ldots \mathcal{D}^{-1} \mathcal{D}\, a^\dagger_- \underbrace{\mathcal{D}^{-1}\, \mathcal{D}}_{\mathbb{1}}\, a^\dagger_- \mathcal{D}^{-1} \mathcal{D} \ldots \mathcal{D}^{-1} \mathcal{D}\, a^\dagger_-}_{\text{from } (a^\dagger_-)^{(j-m)}} \underbrace{\mathcal{D}^{-1}\,\mathcal{D}}_{\mathbb{1}} |0\rangle$$

$$= \left(\mathcal{D}\, a_+^\dagger\, \mathcal{D}^{-1}\right)^{(j+m)} \times \left(\mathcal{D}\, a_-^\dagger\, \mathcal{D}^{-1}\right)^{(j-m)} \times \mathcal{D}|0\rangle \quad \ldots (2)$$

But $\mathcal{D}|0\rangle = |0\rangle$ ...(3), since vacuum (no angular momentum) is rotationally invariant

[Explicitly, $\mathcal{D}(\alpha = 0, \beta, \gamma = 0)$ that we care about $= \exp(-iJ_y\beta/\hbar)$

$$= \mathbb{1} + (-iJ_y\beta/\hbar) + (-iJ_y\beta/\hbar)^2/2! + \cdots$$

with $J_y = \dfrac{J_+ - J_-}{2i}$ is made up of $a_\pm$ (annihilation operators) to the right, which give null ket upon acting on $|0\rangle$ $\Rightarrow$ $\exp(-iJ_y\beta/\hbar)|0\rangle = |0\rangle$

from $\mathbb{1}$ part of $\exp(-iJ_y\beta/\hbar)$]

We need to evaluate (specializing to rotation about y-axis)

$$\mathcal{D}(R)\, a_\pm^\dagger\, \mathcal{D}^{-1}(R) = \exp\left(-\frac{iJ_y\beta}{\hbar}\right) a_\pm^\dagger \exp\left(\frac{iJ_y\beta}{\hbar}\right)$$

Use Baker-Hausdorff lemma:

$$\exp(iG\lambda)\, A\, \exp(-iG\lambda) = A + i\lambda[G, A] + \left(\frac{i^2\lambda^2}{2!}\right)[G,[G,A]] + \cdots$$

with $G \to -J_y/\hbar$; $\lambda \to \beta$; $A \to a_\pm^\dagger$ and

creates spin-up ("inspired" by oscillator model)

$$J_y = (J_+ - J_-)/(2i) = \frac{\hbar}{2i}\left(a_+^\dagger a_- - a_-^\dagger a_+\right)$$

independent spaces

so that $[J_y, a_+^\dagger] = \frac{\hbar}{2i}\left(\cancel{[a_+^\dagger a_-, a_+^\dagger]}_{0} - [a_-^\dagger a_+, a_+^\dagger]\right)$

(1st commutator in series)

$$= \frac{\hbar}{2i} \times -a_-^\dagger[a_+, a_+^\dagger] = -\frac{a_-^\dagger\hbar}{2i} \quad \text{etc...}$$

Similarly, 2nd "nested" commutator ~

$$[J_y, [J_y, a_+^\dagger]] = \left[J_y, -\frac{a_-^\dagger \hbar}{2i}\right] \quad \text{(use 1st result)}$$

$$\propto \left([a_+^\dagger a_-, a_-^\dagger] - [\cancel{a_-^\dagger a_+}, a_-^\dagger]\right) = a_+^\dagger [a_-, a_-^\dagger] = a_+^\dagger \ldots$$

... giving always $a_+^\dagger$ or $a_-^\dagger$

$$\boxed{\begin{aligned} \mathcal{D}(R)\, a_+^\dagger \mathcal{D}^{-1}(R) &= a_+^\dagger \cos\beta/2 + a_-^\dagger \sin\beta/2 \\ \mathcal{D}(R)\, a_-^\dagger \mathcal{D}^{-1}(R) &= a_-^\dagger \cos\frac{\beta}{2} - a_+^\dagger \sin\frac{\beta}{2} \quad \cdots (4) \end{aligned}}$$

[as expected: single spin-up state — vs. operator above — transforms as

$$\mathcal{D}(R)\, a_+^\dagger |0\rangle = \cos\beta/2 \underbrace{a_+^\dagger|0\rangle}_{\text{spin-up}} + \sin\frac{\beta}{2} \underbrace{a_-^\dagger|0\rangle}_{\text{spin-down}}$$

using $\mathcal{D}(R) = d^{1/2} \doteq \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}$, with

matrix form

$$a_+^\dagger|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } a_-^\dagger|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Above $\mathcal{D}(R)\, a_+^\dagger|0\rangle = \mathcal{D}(R)\, a_+^\dagger \overbrace{\mathcal{D}^{-1}(R)\,\mathcal{D}(R)}^{\mathbb{1}}|0\rangle$

$= [\mathcal{D}(R)\, a_+^\dagger \mathcal{D}^{-1}(R)]\,|0\rangle$ ... agreeing with earlier (4) ...]

Plugging (3), (4) into (2) and using binomial expansion

$$(x+y)^N = \sum x^{N-k} y^k \underbrace{{}^N C_k}_{N!/[k!(N-k)!]}$$

gives schematically [for LHS of (1)]

$$\boxed{\mathcal{D}(R)\,|j,m\rangle = (\text{multiple})\ \text{sum over powers of } a_\pm^\dagger \cos\beta/2 \ \&\ a_\pm^\dagger \sin\beta/2}$$

(in detail)

$$= \underbrace{\sum_k \frac{(j+m)!\,\left(a_+^\dagger c_{\beta/2}\right)^{j+m-k} \left(a_-^\dagger s_{\beta/2}\right)^k}{(j+m-k)!\,k!}}_{\text{from } (\mathcal{D} a_+^\dagger \mathcal{D}^{-1})^{(j+m)}}$$

$$\times \underbrace{\sum_\ell \frac{(j-m)!\,\left(-a_+^\dagger s_{\beta/2}\right)^{j-m-\ell} \left(a_-^\dagger c_{\beta/2}\right)^\ell}{(j-m-\ell)!\,\ell!}}_{\text{from } (\mathcal{D} a_-^\dagger \mathcal{D}^{-1})^{(j-m)}} \frac{|0\rangle}{\sqrt{(j+m)!\,(j-m)!}}$$

$s_{\beta/2}$: $\sin\beta/2$; $c_{\beta/2}$: $\cos\beta/2$

"Matching" powers of $a_\pm^\dagger$ on LHS, RHS of (1)...

(Wigner's formula) $\boxed{d^{(j)}_{m'm}(\beta) = \text{sum over powers of } \cos\beta/2 \ \&\ \sin\beta/2}$

In more detail,

$$d^{(j)}_{m'm}(\beta) = \sum_k (-1)^{k+m'-m} \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,k!\,(j-k-m')!\,(k-m+m')!}$$
$$\times \left(\cos \beta/_2\right)^{(2j-2k+m-m')} \left(\sin \beta/_2\right)^{(2k-m+m')}$$

[k summed such that arguments of factorials in denominator are all non-negative]

- See Sakurai Eqs. 3.9.31 onwards for derivation

- Informal HW: check that Wigner's formula for general $d^{(j)}_{m'm}(\beta)$ in Eq. 3.9.33 reproduces $d^{(j=1/2)}_{m'm}$ of Eq. 3.5.52 (middle of Eq. 3.3.21) done in lecture & $d^{(j=1)}_{m'm}$ of Eq. 3.5.57 (problem 3.26 of Sakurai in HW 7.2, done partly in lecture)