Lecture $\boxed{34}$, Nov. 18 (Wed.)

Outline for today (& Fri.)

- So far, angular momenta "added" using angular momentum eigenkets: how new basis (eigenkets of net $\vec{J} = \vec{J}_1 + \vec{J}_2$) related to old basis (eigenkets of $\vec{J}_{1,2}$) by CG coefficients
- Next, addition of angular momenta in terms of rotation operators/matrices: relating these matrix elements in new vs. old basis by CG coefficients
- Onto Schwinger's model: "realize" angular momentum algebra (commutation relations) using (pair of) SHO
  - academic/math interest/curiosity
  - alternate/easier way to obtain rotation matrices

Rotation operators/matrices in new basis related to old/original basis

- Notation: $\mathcal{D}^{(j_1, j_2)}$ are rotation operators in two independent spaces, with bases formed by angular momentum eigenkets with eigenvalues of $|\vec{J}_{1,2}|^2$ being given by $j_{1,2}$ (old basis)
- Recall $\boxed{\text{adding}}$ $\vec{J}_{1,2}$ gives eigenvalues of net $|\vec{J}|^2$: $j(j+1)\hbar^2$, with $j = |j_1 - j_2| \ldots j_1 + j_2$ (new basis)

$\Rightarrow$ $\mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)}$ ["product" of rotations]

= "sum" of rotations given by j in new basis

$$= \mathcal{D}^{\overbrace{(j_1 + j_2)}^{}}{}_{\substack{\\ j_{max}}} \oplus \mathcal{D}^{(j_1+j_2-1)} \oplus \ldots \mathcal{D}^{|j_1 - j_2|}$$

$$= \mathcal{D}^{\underbrace{(j_1+j_2)}_{j_{max}}} \oplus \mathcal{D}^{(j_1+j_2-1)} \oplus \ldots \mathcal{D}^{\underbrace{|j_1-j_2|}_{j_{min}}}$$

$$\begin{bmatrix} \mathcal{D}^{(j_1+j_2)} & 0 & \cdots & 0 \\ 0 & \mathcal{D}^{(j_1+j_2-1)} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \mathcal{D}^{|j_1-j_2|} \end{bmatrix}$$

off-diagonal [each net j rotation is "independent"]

[Recall: in new basis, arbitrary rotation – not characterized by single j – can be "block-diagonalized", with each block corresponding to one j]

– Onto rotation matrix elements: new basis related to old by CG coefficients (since CG coefficients relate old to new base kets used to represent rotation operator)

$\boxed{\text{Claim}}$:

$$\underbrace{\mathcal{D}^{(j_1)}_{m_1 m_1'}(R)\,\mathcal{D}^{(j_2)}_{m_2 m_2'}(R)}_{\text{rotation matrix elements in old/separate } \vec{J}_{1,2} \text{ basis}} = \sum_{j=j_{min}}^{j_{max}} \sum_{m} \sum_{m'}$$

$$\left.\langle j_1 j_2; m_1 m_2 | j_1 j_2; j\, m\rangle \langle j_1 j_2; m_1' m_2' | j_1 j_2; j\, m'\rangle\right\} \text{CG coefficients} \times$$

$$\left.\mathcal{D}^{(j)}_{m m'}(R)\right\} \text{rotation matrix element in new/net } \vec{J} \text{ basis}$$

$\boxed{\text{Proof}}$ We have matrix element of full rotation operator in old basis:

$$\langle \underbrace{j_1 j_2;}_{\substack{\text{drop} \\ \text{for simplicity}}} m_1 m_2 | \mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)} | \underbrace{j_1 j_2;}_{\text{drop...}} m_1' m_2' \rangle$$

$$= \langle m_1 | \otimes \langle m_2 | \mathcal{D}^{(j_1)} \times \mathcal{D}^{(j_2)} | m_1' \rangle \times | m_2' \rangle$$

$$= \langle m_1 | \mathcal{D}^{(j_1)} | m_1' \rangle \langle m_2 | \mathcal{D}^{(j_2)} | m_2' \rangle$$

(= LHS of claim)

$\mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)}$

However, LHS also $= \langle m_1 m_2 | \mathcal{D}(R) | m_1' m_2' \rangle$

$$= \sum_j \sum_m \sum_{j'} \sum_{m'} \langle m_1 m_2 \underbrace{| j m \rangle \langle j m |}_{\mathbb{1}} \mathcal{D}(R) \ldots$$

$$\ldots \underbrace{| j' m' \rangle \langle j' m' |}_{\mathbb{1}} m_1' m_2' \rangle$$

$$= \sum_j \sum_m \sum_{j'} \sum_{m'} \langle m_1 m_2 | j m \rangle \times$$

CG coeffi-cients

use $\langle j' m' | m_1' m_2' \rangle = \langle m_1' m_2' | j' m' \rangle^*$ = (real) $\langle m_1' m_2' | j' m' \rangle$

$$\langle m_1' m_2' | j' m' \rangle \times$$

$$\underbrace{\langle j m | \mathcal{D}(R) | j' m' \rangle}_{\delta_{jj'} \langle j m | \mathcal{D}(R) | j m' \rangle}$$

"collapses" $\sum_{j'}$

cannot connect $j \neq j'$

$$= \sum_{j m m'} \langle m_1 m_2 | j m \rangle \langle m_1' m_2' | j m' \rangle \times \mathcal{D}^{(j)}_{m m'} = \text{RHS of claim}$$

- **Application**: integral of three spherical harmonics (useful for multipole matrix elements in spectroscopy: spherical harmonics are angular part of energy wavefunctions, thus overlap involves integral + related to Wigner-Eckart theorem)

Rough idea: Y_ℓ^m's related to specific $\mathcal{D}$:

So, $\mathcal{D}$'s $\rightarrow$ Y's in above claim gives

$$\int d\Omega\, Y^* \underbrace{\text{product of } 2\, Y\text{'s}} \sim \int d\Omega\, Y^* (\text{single})\, Y \times \text{CG coefficients}$$

Use orthonormality of Y's on RHS to give

$$\boxed{\int Y^3 \sim \text{product of 2 CG coefficients}}$$

Details: $\mathcal{D}^{(\ell)}_{m\,0}(\alpha,\beta,\gamma=0) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^{m\,*}(\theta=\beta,\phi=\alpha)$

In above claim, set $j_{1,2} \rightarrow \ell_{1,2}$ & $\underbrace{m'_{1,2} \rightarrow 0}$:

$\Rightarrow$

$\Rightarrow$ all 3 ϑ's have "m"-index

$$Y_{\ell_1}^{m_1}(\theta,\phi)\, Y_{\ell_2}^{m_2}(\theta,\phi) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}} \sum_{\ell'} \sum_{m'} \sqrt{\frac{4\pi}{2\ell'+1}}\, Y_{\ell'}^{m'}(\theta,\phi) \times$$

$$\langle \ell_1 \ell_2 ; m_1 m_2 | \ell_1 \ell_2 ; \ell\, m \rangle \langle \ell_1 \ell_2 ; \underbrace{0\ 0}_{m'_{1,2}} | \ell_1 \ell_2 ; \ell'\, \underset{\uparrow\, m'}{0} \rangle$$

Then $\int d\Omega\, Y_\ell^{m\,*}(\theta,\phi) \times \ldots$ on both sides, using orthornormality of $Y_\ell^m(\theta,\phi)$ to collapse sums on RHS, gives

$$\int d\Omega\, Y_\ell^{m\,*}(\theta,\phi)\, Y_{\ell_1}^{m_1}(\theta,\phi)\, Y_{\ell_2}^{m_2}(\theta,\phi) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \times$$

$$\underbrace{\langle \ell_1 \ell_2 ; 0\ 0 | \ell_1 \ell_2 ; \ell, 0 \rangle}_{\text{independent of } m_{1,2} \text{ (orientations)}} \times \underbrace{\langle \ell_1 \ell_2 ; m_1 m_2 | \ell_1 \ell_2 ; \ell\, m \rangle}_{\text{CG coefficient for adding } \ell_{1,2} \text{ to give } \ell}$$

Schwinger's oscillator model for angular momentum (section $\boxed{3.9}$)

Motivation: ladder (raising/lowering) operators used in first in SHO and angular momentum $\Rightarrow$ develop analogy between them: another viewpoint/curiosity + easier way to get rotation matrices

- Let's then "connect" two independent/uncoupled oscillators to angular momenta:

Put "$\pm$" labels/subscripts on (single) SHO (usual) relations: $a^{(\dagger)(\text{"dagger"})}_{\pm\ (\text{"plus/minus"})}$ ($a_+^\dagger$ creates of "+" type; a_- annihilates "$-$"...)

(separate) Number operators: $N_\pm = a_\pm^\dagger a_\pm$;

$$[a_\pm, a_\pm^\dagger] = 1;\ [N_\pm, a_\pm] = -a_\pm;\ [N, a^\dagger] = a_\pm^\dagger$$

Operators for different oscillators commute (independent vector spaces):

$$[a_+, a_-^\dagger] = [a_-, a_+^\dagger] = 0$$

$$\Rightarrow [N_+, N_-] = 0 \quad \Rightarrow \text{simultaneous eigenkets:}$$

$$N_\pm \underbrace{|n_+, n_-\rangle}_{\text{new notation}} = n_\pm |n_+, n_-\rangle, \text{ with}$$

$$a_+^\dagger |n_+, n_-\rangle = \sqrt{n_+ + 1}\, |n_+ + 1, n_-\rangle$$

("$-$" type unchanged)

$$a_- |n_+, n_-\rangle = \sqrt{n_-}\, |n_+, n_- - 1\rangle$$

n_+ → unaffected

$$\Rightarrow \text{eigenket of } N_\pm,\ |n_+, n_-\rangle = \frac{(a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-}}{\sqrt{n_+!}\sqrt{n_-!}}$$

$\boxed{\text{acting on}}$ $|0,0\rangle$, where

$$a_\pm \underbrace{|0,0\rangle}_{\text{ground state of both oscillators}} = 0$$

(non-zero energy, even if $n_\pm = 0$)

So far, 2 oscillators separately considered

Next, "combine" them: use "J" notation in anticipation of connection to angular momentum:
just to distinguish from actual angular momentum

$$J_+^{osc} \equiv \hbar\, a_+^\dagger a_- \left(\text{creates "+" type; annihilates "−": both involved} \right)$$

$$J_-^{osc} \equiv \hbar\, a_-^\dagger a_+ \left(= \hbar\, a_+ a_-^\dagger \right)$$

$$J_z^{osc} \equiv \frac{\hbar}{2} \left(N_+ - N_- \right),$$ which

satisfy angular momentum commutation relations (see bottom of next slide)

$$\left[J_z^{osc}, J_\pm^{osc} \right] = \pm \hbar J_\pm^{osc}; \left[J_+^{osc}, J_-^{osc} \right] = 2\hbar J_z^{osc}$$

– Also, $\left| \vec{J}^{osc} \right|^2 \equiv \left(J_z^{osc} \right)^2 + \frac{1}{2} \left(J_+^{osc} J_-^{osc} + J_-^{osc} J_+^{osc} \right)$

$= \frac{\hbar^2}{2} N \left(\frac{N}{2} + 1 \right)$, where $N \equiv N_+ + N_-$

$= a_+^\dagger a_+ + a_-^\dagger a_-$

$\Rightarrow |n_+, n_-\rangle$ is eigenket of J_z^{osc} & $|\vec{J}^{osc}|^2$:

$$J_z^{osc}|n_+, n_-\rangle = \frac{\hbar}{2}(n_+ - n_-)|n_+, n_-\rangle$$

$$|\vec{J}^{osc}|^2|n_+, n_-\rangle = \frac{\hbar^2}{2}\underbrace{(n_+ + n_-)}_{\text{eigenvalue of } N}\left(\frac{n_+ + n_-}{2} + 1\right) \times |n_+, n_-\rangle$$

(Equivalently, $\left[J_z^{osc}, |\vec{J}^{osc}|^2\right] = 0$)

So, $\vec{J}^{osc}$ mathematically behave like usual $\vec{J}$ (physical angular momentum)

Proof of $\vec{J}^{osc}$ commutation relations:

e.g., $\left[J_+^{osc}, J_-^{osc}\right] = \hbar^2\left[a_+^\dagger a_- - a_-^\dagger a_+\right]$

$$= \hbar^2\left(a_+^\dagger a_- a_-^\dagger a_+ - a_-^\dagger a_+ a_+^\dagger a_-\right)$$

(Use $a_i a_i^\dagger = a_i^\dagger a_i + 1$ from $[a_i, a_i^\dagger] = 1$, with $i = \pm$ here)

$$= \hbar^2 \left\{ a_+^\dagger (a_-^\dagger a_- + 1) a_+ - a_-^\dagger (a_+^\dagger a_+ + 1) a_- \right\}$$

$$= \hbar^2 \Big(a_+^\dagger a_-^\dagger a_- a_+ + a_+^\dagger a_+ - \underbrace{a_-^\dagger a_+^\dagger}_{a_+^\dagger a_-^\dagger} \underbrace{a_+ a_-}_{a_- a_+} - a_-^\dagger a_- \Big)$$

(independent)

($\Rightarrow$ 1st, 3rd terms cancel)

$$= \hbar^2 (a_+^\dagger a_+ - a_-^\dagger a_-) = 2\hbar J_z^{osc}$$

... and so on ...